Published by Institute of Physics Publishing for SISSA
Received: *December 11, 2006*
Accepted: *December 14, 2006*
Published: *January 15, 2007*

Merging heterotic orbifolds and K3 compactifications with line bundles

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Abstract: We clarify the relation between six-dimensional Abelian orbifold compactifications of the heterotic string and smooth heterotic $K3$ compactifications with line bundles for both SO(32) and $E_8 \times E_8$ gauge groups. The $T^4/\mathbb{Z}_N$ cases for $N = 2, 3, 4$ are treated exhaustively, and for $N = 6$ some examples are given. While all $T^4/\mathbb{Z}_2$ and nearly all $T^4/\mathbb{Z}_3$ models have a simple smooth match involving one line bundle only, this is only true for some $T^4/\mathbb{Z}_4$ and $T^4/\mathbb{Z}_6$ cases. We comment on possible matchings with more than one line bundle for the remaining cases.
The matching is provided by comparisons of the massless spectra and their anomalies as well as a field theoretic analysis of the blow-ups.

Keywords: Compactification and String Models, Superstring Vacua, Superstrings and Heterotic Strings, Anomalies in Field and String Theories.

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1. Introduction

Orbifolds of the heterotic string have been known for more than twenty years [1], and four dimensional Standard Model building attempts in this corner of the M-theory star have started to evolve soon after its implementation [2] (see [3] for recent results). New interest in the field was then generated by the introduction of the so-called "orbifold-GUT" idea in extensions of the Standard Model including the presence of extra dimensions [4]. Indeed, many of the latest model building attempts were devoted to a string embedding of this kind of field theory models [5]. Moreover, in [6] it was shown that the $\mathbb{Z}_6'$ orbifold limit of $E_8 \times E_8$ is a particularly fertile background to implement MSSM spectra and compute the supersymmetry breaking with a MSSM spectrum's frequency of 10^{-2} compared to 10^{-9} obtained in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ type IIA orientifold background with intersecting branes [7].

On the other hand, $E_8 \times E_8$ heterotic compactifications on Calabi-Yau manifolds with $\mathrm{SU}(n)$ background gauge bundles and GUT spectra have become popular in the last ten years, see e.g. [8] for some early works, and recent constructions of potentially phenomenologically interesting SU(5) and SO(10) GUT spectra with Wilson line breaking to the standard model can be found e.g. in [9] and [10], respectively.

The two avenues to string phenomenology involving singular and smooth heterotic backgrounds have been pursued essentially independently without making any obvious connection.

Contrariwise, in type II compactifications it is known that D-branes in an orbifold background can be treated on equal footing as those in Calabi-Yau backgrounds in the large volume (geometric) regime by explicitly constructing the cycles at the orbifold point, see e.g. [11] for the treatment within Intersecting Brane Worlds, for more references see also [12].

Furthermore, in [13] it was shown that compactifications with D-branes are S-dual to SO(32) heterotic compactifications with generic $\mathrm{U}(n)$ backgrounds. Using $\mathrm{U}(n)$ bundles embedded in $E_8 \times E_8$, a large class of standard model and flipped SU(5) like chiral spectra were constructed subsequently [14].[1]

In this article, we aim at closing the gap between the different model building approaches by matching heterotic SO(32) and $E_8 \times E_8$ orbifold spectra with smooth counter parts which have $\mathrm{U}(1)$ gauge backgrounds. For concreteness, we focus on perturbative supersymmetric Abelian $T^4/\mathbb{Z}_N$ orbifolds and $K3$ compactifications, both without Wilson lines. The $K3$ cases were treated in full generality in [16] including H5-branes. Taking into account H5-branes is also straightforward on the orbifold side as discussed in [17].

This article is organized as follows:

In section 2 we present the framework to compute $T^4/\mathbb{Z}_N$ heterotic orbifold vacua. We reobtain the classification of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ models given in [17], and completely classify the $\mathbb{Z}_4$ models, both in the SO(32) and in the $E_8 \times E_8$ case. For each orbifold we give the complete list of inequivalent shift embeddings,[2] and the massless spectra of the inequivalent models, making large use of the classification strategy described in [19]. We also show some

[1]For more four dimensional heterotic string compactifications with generic (non)-Abelian see also [15].

[2]In the $E_8 \times E_8$ $\mathbb{Z}_4$ case such a result was already given in [18].

examples of $\mathbb{Z}_6$ models. For each model, we compute the field theoretic anomaly polynomial in six dimensions in section 2.4 and verify its 4×4 factorization.

In section 3.1, we present the model building rules of heterotic $K3$ compactifications with arbitrary gauge backgrounds and compare them with the orbifold construction of section 2. We proceed with the analysis of smooth SO(32) heterotic string compactifications in section 3.2, where we find an explicit match with the orbifold models via the coefficients in the anomaly polynomial. This leads to the idea that a $\mathbb{Z}_N$ orbifold shift vector can be directly translated into the smooth embedding of a line bundle L via the relation

$$\frac{1}{N}(1, \ldots, 1, n, 0, \ldots, 0) \to (L, \ldots, L, L^n, 0, \ldots, 0), \tag{1.1}$$

and the second Chern character (instanton number) of L is computed from the coefficients of the anomaly polynomial at the orbifold point. In section 3.2.1, this recipe is demonstrated by explicitly matching several SO(32) orbifold spectra. We apply the same line of reasoning to the $E_8 \times E_8$ compactifications in section 3.3 and show some explicit matches of $E_8 \times E_8$ orbifold spectra in section 3.3.1.

Up to this point, only the second Chern characters of the line bundles have been determined. In section 4 we speculate on the detailed form of such line bundles.

The ansatz for the orbifold / smooth matching is justified further in section 5 via a field theoretical treatment of the blow-up procedure. We explicitly compute the flat directions of the twisted matter potential and show that the blow-up procedure, i.e. switching on a vev along some flat direction, takes care of some seeming mismatch between orbifold and smooth models.

Finally, in section 6 we conclude, and in appendix A we collect some technical details on gauge symmetry breaking in our models.

2. Perturbative $T^4/\mathbb{Z}_N$ orbifolds of the heterotic string

A perturbative $T^4/\mathbb{Z}_N$ heterotic orbifold [1] (see also [20] for the algorithm), in absence of Wilson lines, is completely specified by the action of the orbifold operator in the geometric space and in the gauge bundle. The space-time part of any $\mathbb{Z}_N$ Abelian orbifold action is given by

$$\theta : z^i \to e^{2\pi i v_i} z^i \tag{2.1}$$

with the shift vector

$$\mathbf{v} \equiv (v_1, v_2, \ldots) = \frac{1}{N}(\sigma_1, \sigma_2, \ldots), \tag{2.2}$$

where σ_i are integers subject to the supersymmetry constraint

$$\sum_i \sigma_i = 0 \bmod 2. \tag{2.3}$$

In this article, we focus on the supersymmetric $T^4/\mathbb{Z}_N$ orbifolds with space-time shift vectors

$$\mathbf{v} = \frac{1}{N}(1, -1), \qquad N = 2, 3, 4, 6. \tag{2.4}$$

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The Abelian orbifold action is embedded into the gauge degrees of freedom by another shift vector

$$\mathbf{V} \equiv (V_1, \ldots, V_{16}) = \frac{1}{N}(\Sigma_1, \ldots, \Sigma_{16}), \tag{2.5}$$

with Σ_i integer numbers. These are the so called "vectorial shifts". The embedding is such that a state with weight vector $\mathbf{w}$ transforms under the orbifold action with a phase given by $e^{2\pi i \mathbf{V}\cdot\mathbf{w}}$. In the even order orbifolds of SO(32) string we also take into account "spinorial shifts", i.e. vectors of the form

$$\mathbf{V_S} = \frac{1}{2N}(\Sigma_1, \ldots, \Sigma_{16}) \tag{2.6}$$

with Σ_i odd integers.

Since the rotation involves spinors also in the gauge bundle, the orbifold action has order N only provided that

$$N \sum_i V_i = 0 \bmod 2. \tag{2.7}$$

Moreover, modular invariance of the partition function, or, equivalently, the level matching condition that combines space-time and gauge shifts, requires

$$N \left(\sum_i V_i^2 - \sum_i v_i^2 \right) = 0 \bmod 2. \tag{2.8}$$

Given these conditions we can classify the possible supersymmetric $T^4/\mathbb{Z}_N$ heterotic orbifolds and compute the spectrum of each model.

2.1 SO(32) models

2.1.1 Gauge symmetry breaking and untwisted spectrum

In a $\mathbb{Z}_N$ orbifold the gauge bosons that are left invariant by the orbifold action are those with weight vector $\mathbf{w}$ such that $\mathbf{V}\cdot\mathbf{w} = 0 \bmod 1$. In the SO(32) case, the weight vectors of the gauge bosons contain only two non-zero entries which have values ± 1, $(\pm 1, \pm 1, 0^{14})$. Thus, in the case of a "vectorial" shift vector $\mathbf{V} = (\Sigma_1, \ldots, \Sigma_{16})/N$, an entry Σ_i is equivalent, for what concerns gauge symmetry breaking, to an entry $\Sigma_i + N$ or $-\Sigma_i$.[3] In case N is even, a generic shift vector has the form[4]

$$\mathbf{V} = \frac{1}{N}\left(0^{n_0}, 1^{n_1}, \ldots, \frac{N}{2}^{n_{N/2}}\right) \tag{2.9}$$

and produces the gauge symmetry breaking

$$\mathrm{SO}(32) \to \mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \ldots \times \mathrm{U}(n_{N/2-1}) \times \mathrm{SO}(2n_{N/2}). \tag{2.10}$$

[3] Notice that the replacement $\Sigma_i \to \Sigma_i + N$ in the shift vector of a model, harmless for what concerns its untwisted spectrum, may modify its twisted matter content, and map it into a different (inequivalent) model, as we comment later.

[4] The superscripts denote the number of identical entries in the shift vector.

The untwisted six dimensional spectrum contains a hyper multiplet in each of the following representations ($N \neq 2$ cases)[5]

$$\begin{aligned}
&(\mathbf{2n_0}, \mathbf{n_1}, \mathbf{1}, \ldots \mathbf{1})_{1,0,\ldots,0}, \\
&(\mathbf{1}, \ldots \mathbf{1}, \mathbf{n_{N/2-1}}, \mathbf{2n_{N/2}})_{0,\ldots,0,1}, \\
&(\mathbf{1}, \ldots, \mathbf{n_p}, \ldots, \mathbf{n_q}, \ldots, \mathbf{1})_{\ldots,1,\ldots,1,\ldots} \text{ for } p+q = N-1,\ q > p, \\
&(\mathbf{1}, \ldots, \mathbf{n_p}, \overline{\mathbf{n}}_{\mathbf{p+1}}, \ldots, \mathbf{1})_{\ldots,1,-1,\ldots}.
\end{aligned} \tag{2.11}$$

The $\mathrm{U}(1)$ charges are always zero but for the $\mathrm{U}(n)$ with $\mathbf{n}$ or $\overline{\mathbf{n}}$ representations for which it is $+1$ and -1, respectively. In the $N = 2$ case, the gauge symmetry breaking is to $\mathrm{SO}(2n_0) \times \mathrm{SO}(2n_1)$, with untwisted matter $(\mathbf{2n_0}, \mathbf{2n_1})$.

In case N is odd, instead, shift vector, gauge symmetry breaking and untwisted matter are given by

$$\begin{aligned}
&\mathbf{V} = \frac{1}{N}\left(0^{n_0}, 1^{n_1}, \ldots, \frac{(N-1)}{2}^{n_{(N-1)/2}}\right), \\
&\mathrm{SO}(32) \to \mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \ldots \times \mathrm{U}(n_{(N-1)/2}), \\
&(\mathbf{2n_0}, \mathbf{n_1}, \mathbf{1}, \ldots \mathbf{1})_{1,0,\ldots,0}, \\
&(\mathbf{1}, \ldots, \mathbf{n_p}, \ldots, \mathbf{n_q}, \ldots, \mathbf{1})_{\ldots,1,\ldots,1,\ldots} \text{ for } p+q = N-1,\ q > p, \\
&\left(\mathbf{1}, \ldots, \frac{\mathbf{n_p}(\mathbf{n_p}-\mathbf{1})}{\mathbf{2}}, \ldots, \mathbf{1}\right)_{\ldots,2,\ldots} \text{ for } p = \frac{N-1}{2}, \\
&(\mathbf{1}, \ldots, \mathbf{n_p}, \overline{\mathbf{n}}_{\mathbf{p+1}}, \ldots, \mathbf{1})_{\ldots,1,-1,\ldots}.
\end{aligned} \tag{2.12}$$

For "spinorial" shift vectors, meaningful only in the even N case, the vector, for what concerns the low energy gauge group, can always be brought to the form

$$\mathbf{V_S} = \frac{1}{2N}\left(1^{n_1}, 3^{n_3}, \ldots, (N-1)^{n_{N-1}}\right) \tag{2.13}$$

with gauge symmetry breaking

$$\mathrm{SO}(32) \to \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{N-1}) \tag{2.14}$$

and untwisted matter

$$\begin{aligned}
&\left(\frac{\mathbf{n_1}(\mathbf{n_1}-\mathbf{1})}{\mathbf{2}}, \ldots, \mathbf{1}\right)_{2,0\ldots,0}, \\
&\left(\mathbf{1}, \ldots, \frac{\mathbf{n_{N-1}}(\mathbf{n_{N-1}}-\mathbf{1})}{\mathbf{2}}\right)_{0\ldots,0,2}, \\
&(\mathbf{1}, \ldots, \mathbf{n_p}, \mathbf{1}, \overline{\mathbf{n}}_{\mathbf{p+2}}, \ldots, \mathbf{1})_{\ldots,1,0,-1,\ldots}.
\end{aligned}$$

[5]Multiplets in six dimensions are CPT invariant, i.e. a hyper multiplet labeled by $\mathbf{R}$ contains a complex scalar in the representation $\mathbf{R}$ as well as a complex scalar in the conjugate representation $\overline{\mathbf{R}}$.

#	Gauge Group Shift Vector	Untwisted Matter	Twisted Matter
2a	$\mathrm{SO}(28) \times \mathrm{SU}(2)^2$ $\frac{1}{2}(1^2, 0^{14})$	$(\mathbf{28}, \mathbf{2}, \mathbf{2}) + 4(\mathbf{1}, \mathbf{1}, \mathbf{1})$	$8(\mathbf{28}, \mathbf{1}, \mathbf{2}) + 32(\mathbf{1}, \mathbf{2}, \mathbf{1})$
2b	$\mathrm{SO}(20) \times \mathrm{SO}(12)$ $\frac{1}{2}(1^6, 0^{10})$	$(\mathbf{20}, \mathbf{12}) + 4(\mathbf{1}, \mathbf{1})$	$8(\mathbf{1}, \mathbf{32}_+)$
2c	$\mathrm{SU}(16) \times \mathrm{U}(1)$ $\frac{1}{4}(1^{15}, -3)$	$2(\mathbf{120})_2 + 4(\mathbf{1})_0$	$16(\mathbf{16})_{-3}$
3a	$\mathrm{SO}(28) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ $\frac{1}{3}(1^2, 0^{14})$	$(\mathbf{28}, \mathbf{2})_1 + 2(\mathbf{1}, \mathbf{1})_0 + 1(\mathbf{1}, \mathbf{1})_2$	$9(\mathbf{28}, \mathbf{2})_{-1/3} + 45(\mathbf{1}, \mathbf{1})_{2/3}$ $+18(\mathbf{1}, \mathbf{1})_{4/3}$
3b	$\mathrm{SO}(22) \times \mathrm{SU}(5) \times \mathrm{U}(1)$ $\frac{1}{3}(1^4, 2, 0^{11})$	$(\mathbf{22}, \mathbf{5})_1 + (\mathbf{1}, \mathbf{10})_2 + 2(\mathbf{1}, \mathbf{1})_0$	$9(\mathbf{22}, \mathbf{1})_{5/3} + 9(\mathbf{1}, \mathbf{10})_{-4/3}$ $+18(\mathbf{1}, \mathbf{5})_{-2/3}$
3c	$\mathrm{SO}(16) \times \mathrm{SU}(8) \times \mathrm{U}(1)$ $\frac{1}{3}(1^8, 0^8)$	$(\mathbf{16}, \mathbf{8})_1 + (\mathbf{1}, \mathbf{28})_2 + 2(\mathbf{1}, \mathbf{1})_0$	$9(\mathbf{1}, \mathbf{28})_{-2/3} + 18(\mathbf{1}, \mathbf{1})_{8/3}$
3d	$\mathrm{SO}(10) \times \mathrm{SU}(11) \times \mathrm{U}(1)$ $\frac{1}{3}(1^{10}, 2, 0^5)$	$(\mathbf{10}, \mathbf{11})_1 + (\mathbf{1}, \mathbf{55})_2 + 2(\mathbf{1}, \mathbf{1})_0$	$9(\mathbf{1}, \mathbf{11})_{-8/3} + 9(\overline{\mathbf{16}}, \mathbf{1})_{-11/6}$
3e	$\mathrm{SU}(14) \times \mathrm{SU}(2)^2 \times \mathrm{U}(1)$ $\frac{1}{3}(1^{14}, 0^2)$	$(\mathbf{14}, \mathbf{2}, \mathbf{2})_1 + (\mathbf{91}, \mathbf{1}, \mathbf{1})_2 + 2(\mathbf{1})_0$	$9(\mathbf{1})_{14/3} + 9(\mathbf{14}, \mathbf{2}, \mathbf{1})_{-4/3}$ $+18(\mathbf{1}, \mathbf{1}, \mathbf{2})_{-7/3}$

Table 1: Perturbative SO(32) heterotic orbifold spectra on $T^4/\mathbb{Z}_N$ for $N = 2, 3$.

2.1.2 Twisted matter

In a $\mathbb{Z}_N$ orbifold, extra sectors of *twisted* matter are expected. In the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases we expect the presence of a single twisted sector in each of the orbifold fixed points. In the $\mathbb{Z}_4$ case, instead, we expect two twisted sectors. Given g the generator of $T^4/\mathbb{Z}_4$, there is a g-twisted spectrum in each of the four fixed points of g and a g^2-twisted spectrum in each of the 16 fixed points of g^2. Similarly, in the $T^4/\mathbb{Z}_6$ case we expect three twisted sectors: a g-twisted one in the single fixed point of g, then a g^2-twisted sector in the nine fixed points of g^2 and a g^3-twisted sector in the 16 fixed points of g^3.

We do not give the details of the computation of the twisted spectrum for the $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_3$ models that have been classified in [17]. In the $T^4/\mathbb{Z}_4$ case, instead, we discuss in detail the classification of the models and summarize the twisted spectrum by using the notation and strategy introduced in [19].

2.1.3 $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_3$ models

Modular invariance forces the number of possible $T^4/\mathbb{Z}_2$ models to three, with labels 2a,b,c. Similarly the number of $T^4/\mathbb{Z}_3$ models is only five, with labels 3a - e. Gauge groups and untwisted matter can be computed from the shift vectors as discussed above. The total spectrum, as already shown in [17] except for the U(1) charges,[6] is summarized in table 1.

[6] Model 3b including U(1) charges has been listed before in [21]. Our charge normalization differs and is chosen such that in the untwisted sector the same charges as for smooth $K3$ embeddings with U(4) gauge group occur, i.e. charge 1 for the fundamental and 2 for the antisymmetric representation of SU(5).

$p = n_1 + 4n_2$	$p = 2$	$p = 10$	$p = 18$	$p = 26$
$(\mathbf{1}, \mathbf{1}, \mathbf{2}_{+}^{\mathbf{n_2-1}})_{\frac{\mathbf{n_1}}{\mathbf{4}}}$	24	12	8	4
$(\mathbf{1}, \overline{\mathbf{n}}_\mathbf{1}, \mathbf{2}_{-}^{\mathbf{n_2-1}})_{\frac{\mathbf{n_1-4}}{\mathbf{4}}}$	12	8	4	0
$(\mathbf{2n_0}, \mathbf{1}, \mathbf{2}_{-}^{\mathbf{n_2-1}})_{\frac{\mathbf{n_1}}{\mathbf{4}}}$	8	4	0	0
$\left(\mathbf{1}, \frac{\overline{\mathbf{n}}_\mathbf{1}(\overline{\mathbf{n}}_\mathbf{1}-\mathbf{1})}{\mathbf{2}}, \mathbf{2}_{+}^{\mathbf{n_2-1}}\right)_{\frac{\mathbf{n_1-8}}{\mathbf{4}}}$	8	4	0	0
$(\mathbf{2n_0}, \overline{\mathbf{n}}_\mathbf{1}, \mathbf{2}_{+}^{\mathbf{n_2-1}})_{\frac{\mathbf{n_1-4}}{\mathbf{4}}}$	4	0	0	0

Table 2: In this table we summarize the g-twisted matter content with *vectorial weight* in SO(32) heterotic $T^4/\mathbb{Z}_4$ models with shift vector $\mathbf{V_a} = \left(0^{n_0=16-n_1-n_2}, 1^{n_1}, 2^{n_2}\right)/4$. We show the multiplicity with which each allowed representation of the gauge group enters in a model, the latter being specified by its shift vector $\mathbf{V_a}$ via $p = n_1 + 4n_2$.

2.1.4 $T^4/\mathbb{Z}_4$ models

Classification of the models:

Two shift vectors are equivalent, i.e. produce the same models, if their difference can be written in terms of the weight vectors of the adjoint or spinorial representation of SO(32), up to irrelevant sign flips. This implies that a complete classification is given assuming the following ansatz for the shift vectors

$$\mathbf{V_a} = \frac{1}{4}\left(0^{n_0=16-n_1-n_2}, 1^{n_1}, 2^{n_2}\right), \tag{2.15}$$

$$\mathbf{V_b} = \frac{1}{4}\left(0^{n_0=16-n_1}, 1^{n_1-1}, 3\right). \tag{2.16}$$

Notice that, for $n_2 = 0$ and fixed value of n_1, $\mathbf{V_a}$ and $\mathbf{V_b}$ are equivalent for what concerns the gauge symmetry breaking and the untwisted matter content. Nevertheless, they produce models with different twisted spectrum [22]. The modular invariance condition can be written as $n_1 + 4n_2 = 2 \bmod 8$ for both $\mathbf{V_a}$ and $\mathbf{V_b}$, in the latter case with $n_2 = 0$.

For $\mathbf{V_a}$ the following two options are allowed:

$$n_1 = 2 + 8p_1, \quad n_2 = 2p_2; \tag{2.17}$$

$$n_1 = 6 + 8p_3, \quad n_2 = 2p_4 + 1. \tag{2.18}$$

The requirement $n_1 + n_2 \leq 16$ constrains the possible values of p_i. We have (inequivalent) solutions with $n_1 = 2$, $n_2 = 0, 2, 4, 6$, and we label the corresponding models by 4a,b,c,d; $n_1 = 10$, $n_2 = 0, 2$, with labels 4e,f; $n_1 = 6$, $n_2 = 1, 3, 5$, with labels 4g,h,i; and $n_1 = 14$, $n_2 = 0$, with label 4j.

For $\mathbf{V_b}$ we have two models with $n_1 = 2$ or $n_1 = 10$, 4a' and 4e'. These models have the same gauge group as 4a and 4e, respectively, but have different twisted matter.

For the spinorial shifts, a similar approach can be followed with shift vector

$$\mathbf{V_S} = \frac{1}{8}\left(1^{n_1}, 3^{n_3=16-n_1}\right). \tag{2.19}$$

$q = n_1 + 4n_0$	$q=2$	$q=10$	$q=18$	$q=26$
$(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{(-1)^{n_2}}}, \mathbf{1}, \mathbf{1})_{-\frac{\mathbf{n_1}}{\mathbf{4}}}$	24	12	8	4
$(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{-(-1)^{n_2}}}, \mathbf{n_1}, \mathbf{1})_{-\frac{\mathbf{n_1-4}}{\mathbf{4}}}$	12	8	4	0
$(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{-(-1)^{n_2}}}, \mathbf{1}, \mathbf{2n_2})_{-\frac{\mathbf{n_1}}{\mathbf{4}}}$	8	4	0	0
$\left(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{(-1)^{n_2}}}, \frac{\mathbf{n_1(n_1-1)}}{\mathbf{2}}, \mathbf{1}\right)_{-\frac{\mathbf{n_1-8}}{\mathbf{4}}}$	8	4	0	0
$(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{(-1)^{n_2}}}, \mathbf{n_1}, \mathbf{2n_2})_{-\frac{\mathbf{n_1-4}}{\mathbf{4}}}$	4	0	0	0

Table 3: In this table we summarize the g-twisted matter content with *spinorial weight* vector in SO(32) heterotic $T^4/\mathbb{Z}_4$ models with shift vectors $\mathbf{V_a}$.

In this case a single parameter is present, and the modular invariance condition can be rephrased as $n_1 = 1 + 4p$. Thus, there are four models with labels 4k,l,m,n.

Gauge groups and untwisted matter:

Gauge groups and untwisted matter can be deduced from the general formulae given above. In the vectorial shift case the gauge group is $\mathrm{SO}(2n_0) \times \mathrm{SU}(n_1) \times \mathrm{SO}(2n_2) \times \mathrm{U}(1)$, with untwisted matter

$$(\mathbf{2n_0}, \mathbf{n_1}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{n_1}, \mathbf{2n_2})_1 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_0. \tag{2.20}$$

In the spinorial shift case the gauge group is $\mathrm{SU}(n_1) \times \mathrm{SU}(n_3) \times \mathrm{U}(1)^2$, with untwisted matter given by

$$\left(\frac{\mathbf{n_1(n_1-1)}}{\mathbf{2}}, \mathbf{1}\right)_{2,0} \oplus \left(\mathbf{1}, \frac{\mathbf{n_2(n_2-1)}}{\mathbf{2}}\right)_{0,2} \oplus (\mathbf{n_1}, \overline{\mathbf{n}}_\mathbf{2})_{1,-1} \oplus (\mathbf{1}, \mathbf{1})_{0,0}. \tag{2.21}$$

***g*-Twisted matter:**

The g-twisted matter can be computed by re-quantizing the closed heterotic string with twisted boundary conditions. This boils down to the identification of the vacua of the sector (i.e. in which representation of the gauge group and of the internal holonomy group they are) and to the classification of the excitations that lift them to the zero-mass level (indeed, all the tachyonic vacua are projected out of the spectrum).

There are two possible vacua, one with vectorial weight vector, the other with spinorial weight vector. Given a model with shift vector[7]

$$\mathbf{V_a} = \frac{1}{4}\left(0^{n_0=16-n_1-n_2}, 1^{n_1}, 2^{n_2}\right) \tag{2.22}$$

and unbroken gauge group

$$\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \mathrm{SO}(2n_2), \tag{2.23}$$

[7] We do not discuss here the two models with vector of the form $\mathbf{V_b}$, their massless spectra are directly given in table 7. The models with spinorial shift vector $\mathbf{V_S}$ are discussed later.

		$n_3 =$ 3	7	11	15
U	$\left(\frac{\mathbf{n_1(n_1-1)}}{\mathbf{2}},\mathbf{1}\right)_{2,0}+\left(\mathbf{1},\frac{\mathbf{n_3(n_3-1)}}{\mathbf{2}}\right)_{0,2}+(\mathbf{n_1},\overline{\mathbf{n}}_\mathbf{3})_{1,-1}+2\,(\mathbf{1},\mathbf{1})_{0,0}$				
T	$n_3 =$	3	7	11	15
	$(\mathbf{1},\mathbf{1})_{\frac{n_1}{8},\frac{3n_3}{8}}$	12	8	4	0
	$\left(\mathbf{1},\frac{\overline{\mathbf{n}}_\mathbf{3}(\overline{\mathbf{n}}_\mathbf{3}-\mathbf{1})}{\mathbf{2}}\right)_{\frac{n_1}{8},\frac{3n_3-16}{8}}$	8	4	0	0
	$(\overline{\mathbf{n}}_\mathbf{1},\overline{\mathbf{n}}_\mathbf{3})_{\frac{n_1-8}{8},\frac{3n_3-8}{8}}$	4	0	0	0
	$(\mathbf{1},\overline{\mathbf{n}}_\mathbf{3})_{-\frac{3n_1}{8},-\frac{n_3+8}{8}}$	0	0	0	4
	$\left(\frac{\mathbf{n_1(n_1-1)(n_1-2)}}{\mathbf{6}},\mathbf{1}\right)_{-\frac{3n_1-24}{8},-\frac{n_3}{8}}$	0	0	4	8
	$(\mathbf{1},\mathbf{n_3})_{-\frac{3n_1}{8},-\frac{n_3-8}{8}}$	0	0	4	8
	$(\mathbf{n_1},\mathbf{1})_{-\frac{3n_1-8}{8},-\frac{n_3}{8}}$	0	4	8	12
T^2	$6\,(\overline{\mathbf{n}}_\mathbf{1},\mathbf{1})_{\frac{n_1-4}{4},-\frac{n_3}{4}}+10\,(\mathbf{1},\mathbf{n_3})_{\frac{n_1}{4},-\frac{n_3-4}{4}}$				

Table 4: Matter content of SO(32) heterotic models on $T^4/\mathbb{Z}_4$ with spinorial shift vector $\mathbf{V_S} = \left(1^{n_1=16-n_3},3^{n_3}\right)/8$. The U entry summarizes the untwisted spectrum. The T table summarizes the g-twisted spectrum giving the multiplicity with which each allowed representation of the gauge group enters in a model, the latter being specified by its shift vector $\mathbf{V_S}$ via n_3. The T^2 line summarizes the g^2-twisted spectrum.

n_1	$n_1=2$	$n_1=6$	$n_1=10$	$n_1=14$
$(\mathbf{1},\mathbf{2}^{\mathbf{n_1-1}}_{\mathbf{(-1)^{n_2}}})$	4	1	0	0
$(\mathbf{2n_0}+\mathbf{2n_2},\mathbf{2}^{\mathbf{n_1-1}}_{\mathbf{-(-1)^{n_2}}})$	1	0	0	0
$(\mathbf{2}^{\mathbf{n_0+n_2-1}}_{\mathbf{-(-1)^{n_2}}},\mathbf{2n_1})$	0	0	0	1
$(\mathbf{2}^{\mathbf{n_0+n_2-1}}_{\mathbf{(-1)^{n_2}}},\mathbf{1})$	0	0	1	4

Table 5: The multiplicity of a given representation of the gauge group in a model. The gauge group is specified by n_1 for the orbifold $T^4/\mathbb{Z}_2$, which is relevant as g^2 twisted sector in the $T^4/\mathbb{Z}_4$ orbifold (more in general, as g^N twisted sector in any $T^4/\mathbb{Z}_{2N}$ orbifold).

the two vacua are in the $(\mathbf{1},\mathbf{1},\mathbf{2}^{\mathbf{n_2-1}}_{+})_{\frac{\mathbf{n_1}}{\mathbf{4}}}$ and in the $(\mathbf{2}^{\mathbf{n_0-1}}_{\mathbf{(-1)^{n_2}}},\mathbf{1},\mathbf{1})_{-\frac{\mathbf{n_1}}{\mathbf{4}}}$ representations, respectively.

For each of the two vacua, the excitations can be switched on among two classes: On the one hand they can change the gauge representation of the vacuum, and then we call them "gauge" excitations. On the other hand they can change the holonomy representation of the vacuum, if we consider excitations due to the four X^m target space bosons along compact

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$n_1 = 2$		reduced states	$\mathbb{Z}_4$ phase
$(\mathbf{1},\mathbf{2}_+)$	$\rightarrow$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{1}}$	$(-1)^{n_2}\times e^{\pi i/2}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{-\mathbf{1}}$	$(-1)^{n_2}\times e^{-\pi i/2}$
$(\mathbf{2n_0}+\mathbf{2n_2},\mathbf{2}_-)$	$\rightarrow$	$(\mathbf{2n_0},\mathbf{2},\mathbf{1})_{\mathbf{0}}$	$(-1)^{n_2}$
		$(\mathbf{1},\mathbf{2},\mathbf{2n_2})_{\mathbf{0}}$	$-(-1)^{n_2}$
$n_1 = 6$		reduced states	$\mathbb{Z}_4$ phase
$(\mathbf{1},\mathbf{2}^{\mathbf{5}}_-)$	$\rightarrow$	$(\mathbf{1},\overline{\mathbf{6}},\mathbf{1})_{\mathbf{2}}$	$-(-1)^{\frac{n_2+1}{2}}$
		$(\mathbf{1},\mathbf{20},\mathbf{1})_{\mathbf{0}}$	$(-1)^{\frac{n_2+1}{2}}$
		$(\mathbf{1},\mathbf{6},\mathbf{1})_{-\mathbf{2}}$	$-(-1)^{\frac{n_2+1}{2}}$
$n_1 = 10$		reduced states	$\mathbb{Z}_4$ phase
$(\mathbf{2}^{\mathbf{5}}_+,\mathbf{1})$	$\rightarrow$	$(\mathbf{2}_+^{\mathbf{n_0-1}},\mathbf{1},\mathbf{2}_+^{\mathbf{n_2-1}})_{\mathbf{0}}$	-1
		$(\mathbf{2}_-^{\mathbf{n_0-1}},\mathbf{1},\mathbf{2}_-^{\mathbf{n_2-1}})_{\mathbf{0}}$	$+1$
$n_1 = 14$ $(n_2 = 1)$		reduced states	$\mathbb{Z}_4$ phase
$(\mathbf{2}_+,\mathbf{28})$	$\rightarrow$	$(\mathbf{1},\mathbf{14},\mathbf{1})_{\frac{1}{2},\mathbf{1},\frac{1}{2}}$	-1
		$(\mathbf{1},\overline{\mathbf{14}},\mathbf{1})_{-\frac{1}{2},-\mathbf{1},-\frac{1}{2}}$	-1
		$(\mathbf{1},\overline{\mathbf{14}},\mathbf{1})_{\frac{1}{2},-\mathbf{1},\frac{1}{2}}$	$+1$
		$(\mathbf{1},\mathbf{14},\mathbf{1})_{-\frac{1}{2},\mathbf{1},-\frac{1}{2}}$	$+1$
$(\mathbf{2}_-,\mathbf{1})$	$\rightarrow$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{\frac{1}{2},\mathbf{0},-\frac{1}{2}}$	$e^{-\pi i/2}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{-\frac{1}{2},\mathbf{0},\frac{1}{2}}$	$e^{\pi i/2}$

Table 6: The $\mathbb{Z}_4$ reduction of the $T^4/\mathbb{Z}_2$ twisted states. Each state should be multiplied by the multiplicity given in the table 5, remembering that a multiplicity 1 brings no extra bosonic phases, while a multiplicity 4 must be split into a 2 with $e^{\pi i/2}$ phase and a 2 with $e^{-\pi i/2}$ phase. The states with global phase $+1$ then receive a $10/2$ multiplicity from the degeneracy of the fixed points, those with phase -1 receive instead a $6/2$ multiplicity.

directions, and then we call them “bosonic” excitations. The excitations of the second class are “visible” from the six dimensional perspective since they change the degeneration of the corresponding twisted state. Both kinds of excitations produce a mass lift, and in a model each gauge excitation is matched with a bosonic excitation to produce a massless state. Since no negative lifting is present, only a few excitations that do not overclose the zero mass condition are allowed. All others produce massive string modes.

Thus, for each vacuum we can just list the “allowed” excited gauge representations, and the allowed multiplicities coming from the bosonic excitations. Then, each representation is present in a model with the multiplicity such that the state is massless. In case the state is always massive, we can just think that the multiplicity is zero. The matching depends on the energy of the vacuum, given by the shift vector $\mathbf{V_a}$ via the numbers n_0, n_1 and n_2.

In our specific case, the possible representations due to the vacuum with vectorial weight are listed in table 2, those due to the vacuum with spinorial weight are listed in table 3. The possible degeneracies due to oscillators are $1, 2, 3, 6$, that must be multiplied by 4, the number of fixed points. In tables 2 and 3 we also resume the matching between degeneracies and representations in the models: the degeneracy with which a representation enters in a model is given as a function of the characteristic numbers n_0, n_1 and n_2 of the model.

As last remarks, we mention some caveats that should be taken into account when reading the tables. In case $p = n_1 + 4n_2 > 14$, the related column is not listed in table 2, since all the entries are 0. Similarly for $q = n_1 + 4n_0 > 14$ in table 3.

In case $n_0 = 1$ and/or $n_2 = 1$ the gauge group SO$(2n_0)$ and/or SO$(2n_2)$ reduces to two/one extra U(1) factor(s), and extra U(1) charges are present. The latter are as follows. The $\mathbf{2_\pm^{n_i-1}}$ representation, a spinorial representation of SO$(2n_i)$ with $\pm$ chirality, shrinks to a singlet of U(1) with charge $\pm1/2$; the $\mathbf{2n_i}$ representation instead shrinks to two singlets with charges ±1. In case $n_i = 0$ $(i = 0, 2)$, the representations including $\mathbf{2n_i}$ or $\mathbf{1_i}$ or $\mathbf{2_+^{n_i-1}}$ are still present, but the $\mathbf{2n_i}$ or $\mathbf{1_i}$ or $\mathbf{2_+^{n_i-1}}$ entry, related to an inexistent SO(0) group, drops. The representations including $\mathbf{2_-^{n_i-1}}$ are instead removed from the spectrum by the GSO projection. As an example of the latter caveat, consider the case $n_1 = 2$, $n_2 = 0$ ("standard embedding"). From table 2 we read off the matter content with vectorial weight from the column with $p = n_1 + 4n_2 = 2$. Since $n_2 = 0$ we have to keep all the states in spinorial representations with + chirality and remove those with $-$ chirality. Thus we have the following matter content: (i) 24 multiplets in the $(\mathbf{1}, \mathbf{1})_{\frac{1}{2}}$, (ii) eight multiplets in the $(\mathbf{1}, \mathbf{1})_{-\frac{3}{2}}$ and (iii) four in the $(\mathbf{28}, \mathbf{2})_{-\frac{1}{2}}$ representations of the gauge group SO(28) $\times$ SU(2) $\times$ U(1). From table 3, instead, we get no extra state, since $q = n_1 + 4n_0 > 14$.

In the case of the spinorial shifts we use the same approach as explained above. The results are listed in table 4.

g^2-twisted matter:

The second twisted sector is built starting from $2\mathbf{V_a} = \frac{1}{4}(0^{n_0}, 2^{n_1}, 4^{n_2}) \sim \frac{1}{2}(0^{n_0+n_2}, 1^{n_1})$, i.e. from the twisted sector of the $T^4/\mathbb{Z}_2$ orbifold. Given this, we have to orbifold such a sector by projecting out the states that are not invariant under the $\mathbb{Z}_4$ operator. We can split the computation into two steps: In the first we compute the twisted sector of $T^4/\mathbb{Z}_2$, summarized in table 5, then we project it. The SO(32) gauge symmetry is broken by $\mathbb{Z}_2$ to SO$(2n_0 + 2n_2) \times$ SO$(2n_1)$, and thus, before the $\mathbb{Z}_4$ projection, the twisted states are organized in multiplets of such a symmetry. We have a multiplicity of one for states without bosonic excitations, and of four for those with an excitation. On top of this, the multiplicity from the fixed point degeneracy, 16, should be added. We do not include the latter in table 5. The table is built as previously, the characteristic number is now n_1. We resume in it all states. The space-time multiplicity is, as in the g^1-twisted sector, just a factor of two, but in this case we do not have a g^3-twisted sector rising such a multiplicity to four, i.e. the multiplicity of a full hyper multiplet. Then, when reading table 5, we have to remember the factor 16 from the fixed points, but also a factor 1/2 necessary in order to correctly count the number of hyper multiplets.

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The second step is the orbifold projection including the $\mathbb{Z}_4$ phases of the obtained states. We summarize the reduction of the states in table 6. Notice that the phase must be combined with the one coming from the bosonic excitations; it is $+1$ in case of no excitation, and it is $e^{\pi i/2}$ for a doublet of the four excited states and $e^{-\pi i/2}$ for the other doublet. The total phase is always either $+1$ or -1. In case the phase is $+1$ the states receive an extra 10/2=5 multiplicity from the fixed point degeneracy, if instead the phase is -1 the multiplicity is 6/2=3.

In the case of the spinorial shifts we use the same approach as explained above, the results are listed in table 4.

The complete massless spectra for heterotic SO(32) orbifolds on $T^4/\mathbb{Z}_4$ are given in table 7 for the vectorial and table 8 for the spinorial shifts.

2.2 $E_8 \times E_8$ models

2.2.1 $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_3$ models

These models have been classified, except for the U(1) charge assignments, in [17]. We summarize them in table 9. The standard embeddings IIa, IIIa (with different U(1) charge normalization) as well as the cases IIIc, IIIe were already presented in [21]. Model IIIe is also discussed in some detail in [23], IIId has been computed before in [24].

2.2.2 $T^4/\mathbb{Z}_4$ models

The inequivalent shift vectors for this orbifold background have been classified in [18] as well as the gauge groups, and the instanton numbers (k_1, k_2) with $k_1 + k_2 = 24$ have been computed for each case. We briefly review the classification here and give some of the details of the computation of the massless spectra - which is done here for the first time - for each shift vector. The results are summarized in table 11.

Gauge symmetry breaking and untwisted matter We can consider each E_8 factor separately. Thus, we split the shift vector into two subvectors with eight entries each. As discussed in [18], the possible subvectors are only ten, listed in table 10.

The gauge symmetry breaking can then be studied by considering first the subgroup SO(16) of E_8. The vectors listed in table 10 are all of the form

$$V_{E_8} = \frac{1}{4}\left(0^{n_0}, 1^{n_1}, -1^{n_{-1}}, 2^{n_2}, 3^{n_3}, 4^{n_4}\right), \tag{2.24}$$

and the breaking of SO(16) is $\mathrm{SO}(16) \to \mathrm{SO}(2n_0 + 2n_4) \times \mathrm{U}(n_1 + n_{-1} + n_3) \times \mathrm{SO}(2n_2)$.

Given the breaking of the SO(16) subgroup, we pass to its enhancement to E_8, due to the presence of gauge bosons in the **128** representation of SO(16). In case some of the states in the **128** representation are left invariant by the orbifold action, the gauge group is enhanced, as shown in table 10.

The untwisted matter can then be obtained as in the SO(32) case, with the caveat that now also the **128** representation is present, and that, in presence of a gauge enhancement, the untwisted states are rearranged in multiplets of the enhanced gauge group, as shown in table 10.

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# & Gauge group		Matter
4a	U	$(\mathbf{28},\mathbf{2})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(28)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	T	$24\ (\mathbf{1},\mathbf{1})_{\frac{1}{2}} + 8\ (\mathbf{1},\mathbf{1})_{-\frac{3}{2}} + 4\ (\mathbf{28},\mathbf{2})_{-\frac{1}{2}}$
(s. e.) $\frac{1}{4}\left(0^{14},1^2\right)$	T^2	$32\ (\mathbf{1},\mathbf{1})_{\mathbf{1}} + 5\ (\mathbf{28},\mathbf{2})_{\mathbf{0}}$
4a'	U	$(\mathbf{28},\mathbf{2})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(28)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	T	$12\ (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} + 4\ (\mathbf{1},\mathbf{2})_{\frac{3}{2}} + 8\ (\mathbf{28},\mathbf{1})_{\frac{1}{2}}$
$\frac{1}{4}\left(0^{14},1,3\right)$	T^2	$32\ (\mathbf{1},\mathbf{1})_{\mathbf{1}} + 5\ (\mathbf{28},\mathbf{2})_{\mathbf{0}}$
4b	U	$(\mathbf{1},\mathbf{2},\mathbf{4})_{\mathbf{1}} + (\mathbf{24},\mathbf{2},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(24)\times\mathrm{SU}(2)\times\mathrm{SO}(4)\times\mathrm{U}(1)$	T	$12(\mathbf{1},\mathbf{1},\mathbf{2}_+)_{\frac{1}{2}} + 8(\mathbf{1},\mathbf{2},\mathbf{2}_-)_{-\frac{1}{2}} +$ $4(\mathbf{24},\mathbf{1},\mathbf{2}_-)_{\frac{1}{2}} + 4(\mathbf{1},\mathbf{1},\mathbf{2}_+)_{-\frac{3}{2}}$
$\frac{1}{4}\left(0^{12},1^2,2^2\right)$	T^2	$32\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{1}} + 5\ (\mathbf{1},\mathbf{2},\mathbf{4})_{\mathbf{0}} + 3\ (\mathbf{24},\mathbf{2},\mathbf{1})_{\mathbf{0}}$
4c	U	$(\mathbf{1},\mathbf{2},\mathbf{8})_{\mathbf{1}} + (\mathbf{20},\mathbf{2},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(20)\times\mathrm{SU}(2)\times\mathrm{SO}(8)\times\mathrm{U}(1)$	T	$8\ (\mathbf{1},\mathbf{1},\mathbf{8}_+)_{\frac{1}{2}} + 4\ (\mathbf{1},\mathbf{2},\mathbf{8}_-)_{-\frac{1}{2}}$
$\frac{1}{4}\left(0^{10},1^2,2^4\right)$	T^2	$32\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{1}} + 3\ (\mathbf{1},\mathbf{2},\mathbf{8})_{\mathbf{0}} + 5\ (\mathbf{20},\mathbf{2},\mathbf{1})_{\mathbf{0}}$
4d	U	$(\mathbf{1},\mathbf{2},\mathbf{12})_{\mathbf{1}} + (\mathbf{16},\mathbf{2},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(16)\times\mathrm{SU}(2)\times\mathrm{SO}(12)\times\mathrm{U}(1)$	T	$4\ (\mathbf{1},\mathbf{1},\mathbf{32}_+)_{\frac{1}{2}}$
$\frac{1}{4}\left(0^{8},1^2,2^6\right)$	T^2	$32\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{1}} + 5\ (\mathbf{1},\mathbf{2},\mathbf{12})_{\mathbf{0}} + 3\ (\mathbf{16},\mathbf{2},\mathbf{1})_{\mathbf{0}}$
4e	U	$(\mathbf{12},\mathbf{10})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(12)\times\mathrm{SU}(10)\times\mathrm{U}(1)$	T	$12\ (\mathbf{1},\mathbf{1})_{\frac{5}{2}} + 4\ (\mathbf{1},\overline{\mathbf{45}})_{\frac{1}{2}}$
$\frac{1}{4}\left(0^{6},1^{10}\right)$	T^2	$3\ (\mathbf{32}_+,\mathbf{1})_{\mathbf{0}}$
4e'	U	$(\mathbf{12},\mathbf{10})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(12)\times\mathrm{SU}(10)\times\mathrm{U}(1)$	T	$8\ (\mathbf{1},\overline{\mathbf{10}})_{\frac{3}{2}} + 4\ (\mathbf{12},\mathbf{1})_{\frac{5}{2}}$
$\frac{1}{4}\left(0^{6},1^{9},3\right)$	T^2	$5\ (\mathbf{32}_+,\mathbf{1})_{\mathbf{0}}$
4f	U	$(\mathbf{1},\mathbf{10},\mathbf{4})_{\mathbf{1}} + (\mathbf{8},\mathbf{10},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(8)\times\mathrm{SU}(10)\times\mathrm{SO}(4)\times\mathrm{U}(1)$	T	$8\ (\mathbf{1},\mathbf{1},\mathbf{2}_+)_{\frac{5}{2}} + 4\ (\mathbf{1},\overline{\mathbf{10}},\mathbf{2}_-)_{\frac{3}{2}} + 4\ (\mathbf{8}_+,\mathbf{1},\mathbf{1})_{-\frac{5}{2}}$
$\frac{1}{4}\left(0^{4},1^{10},2^2\right)$	T^2	$3\ (\mathbf{8}_+,\mathbf{1},\mathbf{2}_+)_{\mathbf{0}} + 5\ (\mathbf{8}_-,\mathbf{1},\mathbf{2}_-)_{\mathbf{0}}$
4g	U	$(\mathbf{1},\mathbf{6})_{\mathbf{1},\mathbf{1}} + (\mathbf{1},\mathbf{6})_{\mathbf{1},-\mathbf{1}} + (\mathbf{18},\mathbf{6})_{\mathbf{1},\mathbf{0}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0},\mathbf{0}}$
$\mathrm{SO}(18)\times\mathrm{SU}(6)\times\mathrm{U}(1)^2$	T	$12\ (\mathbf{1},\mathbf{1})_{\frac{3}{2},\frac{1}{2}} + 8\ (\mathbf{1},\overline{\mathbf{6}})_{\frac{1}{2},-\frac{1}{2}} + 4\ (\mathbf{18},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} + 4\ (\mathbf{1},\overline{\mathbf{15}})_{-\frac{1}{2},\frac{1}{2}}$
$\frac{1}{4}\left(0^{9},1^{6},2\right)$	T^2	$3\ (\mathbf{1},\mathbf{20})_{\mathbf{0},\mathbf{0}} + 10\ (\mathbf{1},\mathbf{6})_{-\mathbf{2},\mathbf{0}}$
4h	U	$(\mathbf{1},\mathbf{6},\mathbf{6})_{\mathbf{1}} + (\mathbf{14},\mathbf{6},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(14)\times\mathrm{SU}(6)\times\mathrm{SO}(6)\times\mathrm{U}(1)$	T	$8\ (\mathbf{1},\mathbf{1},\mathbf{4})_{\frac{3}{2}} + 4\ (\mathbf{1},\overline{\mathbf{6}},\overline{\mathbf{4}})_{\frac{1}{2}}$
$\frac{1}{4}\left(0^{7},1^{6},2^3\right)$	T^2	$5\ (\mathbf{1},\mathbf{20},\mathbf{1})_{\mathbf{0}} + 6\ (\mathbf{1},\mathbf{6},\mathbf{1})_{-\mathbf{2}}$
4i	U	$(\mathbf{1},\mathbf{6},\mathbf{10})_{\mathbf{1}} + (\mathbf{10},\mathbf{6},\mathbf{1})_{\mathbf{1}} + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_{\mathbf{0}}$
$\mathrm{SO}(10)\times\mathrm{SU}(6)\times\mathrm{SO}(10)\times\mathrm{U}(1)$	T	$4\ (\mathbf{1},\mathbf{1},\mathbf{16})_{\frac{3}{2}} + 4\ (\overline{\mathbf{16}},\mathbf{1},\mathbf{1})_{-\frac{3}{2}}$
$\frac{1}{4}\left(0^{5},1^{6},2^5\right)$	T^2	$3\ (\mathbf{1},\mathbf{20},\mathbf{1})_{\mathbf{0}} + 10\ (\mathbf{1},\mathbf{6},\mathbf{1})_{-\mathbf{2}}$
4j	U	$(\mathbf{14})_{\mathbf{0},\mathbf{1},\mathbf{1}} + (\mathbf{14})_{\mathbf{0},\mathbf{1},-\mathbf{1}} + (\mathbf{14})_{\mathbf{1},\mathbf{1},\mathbf{0}} + (\mathbf{14})_{-\mathbf{1},\mathbf{1},\mathbf{0}} + 2\ (\mathbf{1})_{\mathbf{0},\mathbf{0},\mathbf{0}}$
$\mathrm{SU}(14)\times\mathrm{U}(1)^3$	T	$8\ (\mathbf{1})_{\mathbf{0},\frac{7}{2},\frac{1}{2}} + 8\ (\mathbf{1})_{-\frac{1}{2},-\frac{7}{2},\mathbf{0}} + 4\ (\overline{\mathbf{14}})_{\mathbf{0},\frac{5}{2},-\frac{1}{2}} + 4\ (\mathbf{14})_{\frac{1}{2},-\frac{5}{2},\mathbf{0}}$
$\frac{1}{4}\left(0,1^{14},2\right)$	T^2	$32(\mathbf{1})_{\frac{1}{2},\mathbf{0},-\frac{1}{2}} + 10(\mathbf{14})_{-\frac{1}{2},\mathbf{1},-\frac{1}{2}} + 6(\mathbf{14})_{\frac{1}{2},\mathbf{1},\frac{1}{2}}$

Table 7: Gauge group and massless matter of SO(32) heterotic models on $T^4/\mathbb{Z}_4$ with vectorial shift vector.

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# & Gauge group		Matter
4k	U	$(\mathbf{105})_{\mathbf{0,2}} + (\overline{\mathbf{15}})_{\mathbf{1,-1}} + 2\ (\mathbf{1})_{\mathbf{0,0}}$
$\mathrm{SU}(15) \times \mathrm{U}(1)^2$	T	$12\ (\mathbf{1})_{\frac{5}{8},-\frac{15}{8}} + 8\ (\mathbf{15})_{-\frac{3}{8},-\frac{7}{8}} + 4\ (\overline{\mathbf{15}})_{-\frac{3}{8},-\frac{23}{8}}$
$\frac{1}{8}\left(1, 3^{15}\right)$	T^2	$6\ (\mathbf{1})_{-\frac{3}{4},-\frac{15}{4}} + 10\ (\mathbf{15})_{\frac{1}{4},-\frac{11}{4}}$
4l	U	$(\mathbf{10},\mathbf{1})_{\mathbf{2,0}} + (\mathbf{1},\mathbf{55})_{\mathbf{0,2}} + (\mathbf{5},\overline{\mathbf{11}})_{\mathbf{1,-1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0,0}}$
$\mathrm{SU}(5) \times \mathrm{SU}(11) \times \mathrm{U}(1)^2$	T	$8\ (\mathbf{5},\mathbf{1})_{-\frac{7}{8},-\frac{11}{8}} + 4\ (\mathbf{1},\mathbf{11})_{-\frac{15}{8},-\frac{3}{8}} + 4\ (\overline{\mathbf{10}},\mathbf{1})_{\frac{9}{8},-\frac{11}{8}} + 4\ (\mathbf{1},\mathbf{1})_{\frac{5}{8},\frac{33}{8}}$
$\frac{1}{8}\left(1^5, 3^{11}\right)$	T^2	$6\ (\overline{\mathbf{5}},\mathbf{1})_{\frac{1}{4},-\frac{11}{4}} + 10\ (\mathbf{1},\mathbf{11})_{\frac{5}{4},-\frac{7}{4}}$
4m	U	$(\mathbf{36},\mathbf{1})_{\mathbf{2,0}} + (\mathbf{1},\mathbf{21})_{\mathbf{0,2}} + (\mathbf{9},\overline{\mathbf{7}})_{\mathbf{1,-1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0,0}}$
$\mathrm{SU}(9) \times \mathrm{SU}(7) \times \mathrm{U}(1)^2$	T	$4\ (\mathbf{9},\mathbf{1})_{-\frac{19}{8},-\frac{7}{8}} + 4\ (\mathbf{1},\overline{\mathbf{21}})_{\frac{9}{8},\frac{5}{8}} + 8\ (\mathbf{1},\mathbf{1})_{\frac{9}{8},\frac{21}{8}}$
$\frac{1}{8}\left(1^9, 3^7\right)$	T^2	$6\ (\overline{\mathbf{9}},\mathbf{1})_{\frac{5}{4},-\frac{7}{4}} + 10\ (\mathbf{1},\mathbf{7})_{\frac{9}{4},-\frac{3}{4}}$
4n	U	$(\mathbf{78},\mathbf{1})_{\mathbf{2,0}} + (\mathbf{1},\overline{\mathbf{3}})_{\mathbf{0,2}} + (\mathbf{13},\overline{\mathbf{3}})_{\mathbf{1,-1}} + 2\ (\mathbf{1},\mathbf{1})_{\mathbf{0,0}}$
$\mathrm{SU}(13) \times \mathrm{SU}(3) \times \mathrm{U}(1)^2$	T	$4\ (\overline{\mathbf{13}},\overline{\mathbf{3}})_{\frac{5}{8},\frac{1}{8}} + 8\ (\mathbf{1},\mathbf{3})_{\frac{13}{8},-\frac{7}{8}} + 12\ (\mathbf{1},\mathbf{1})_{\frac{13}{8},\frac{9}{8}}$
$\frac{1}{8}\left(1^{13}, 3^3\right)$	T^2	$6\ (\overline{\mathbf{13}},\mathbf{1})_{\frac{9}{4},-\frac{3}{4}} + 10\ (\mathbf{1},\mathbf{3})_{\frac{13}{4},\frac{1}{4}}$

Table 8: SO(32) heterotic models on $T^4/\mathbb{Z}_4$ with spinorial shift vector.

#	Group Shift	Untwisted Matter	Twisted Matter
IIa	$E_7 \times \mathrm{SU}(2) \times E_8$ $\frac{1}{2}(1^2, 0^6) \times (0^8)$	$(\mathbf{56},\mathbf{2}) + 4(\mathbf{1},\mathbf{1})$	$8(\mathbf{56},\mathbf{1}) + 32(\mathbf{1},\mathbf{2})$
IIb	$\mathrm{SO}(16) \times E_7 \times \mathrm{SU}(2)$ $\frac{1}{2}(1, 0^7) \times (1^2, 0^6)$	$(\mathbf{1};\mathbf{56},\mathbf{2}) + (\mathbf{128};\mathbf{1},\mathbf{1}) + 4(\mathbf{1})$	$8(\mathbf{16},\mathbf{1},\mathbf{2})$
IIIa	$E_7 \times \mathrm{U}(1) \times E_8$ $\frac{1}{3}(1^2, 0^6) \times (0^8)$	$(\mathbf{56},\mathbf{1})_1 + (\mathbf{1},\mathbf{1})_2 + 2(\mathbf{1},\mathbf{1})_0$	$9(\mathbf{56},\mathbf{1})_{\frac{1}{3}} + 45(\mathbf{1},\mathbf{1})_{\frac{2}{3}}$ $+18(\mathbf{1},\mathbf{1})_{\frac{4}{3}}$
IIIb	$\mathrm{SO}(14)^2 \times \mathrm{U}(1)^2$ $\frac{1}{3}(2, 0^7) \times \frac{1}{3}(2, 0^7)$	$(\mathbf{14},\mathbf{1})_{1,0} + (\mathbf{1},\mathbf{14})_{0,1} + (\mathbf{64},\mathbf{1})_{\frac{1}{2},0}$ $+(\mathbf{1},\mathbf{64})_{0,\frac{1}{2}} + 2(\mathbf{1},\mathbf{1})_{0,0}$	$9(\mathbf{14},\mathbf{1})_{-\frac{1}{3},\frac{2}{3}} + 9(\mathbf{1},\mathbf{14})_{\frac{2}{3},-\frac{1}{3}}$ $+18(\mathbf{1},\mathbf{1})_{\frac{2}{3},\frac{2}{3}}$
IIIc	$\mathrm{SU}(9) \times E_8$ $\frac{1}{3}(1^4, 2, 0^3) \times (0^8)$	$(\mathbf{84},\mathbf{1}) + 2(\mathbf{1},\mathbf{1})$	$9(\mathbf{36},\mathbf{1}) + 18(\mathbf{9},\mathbf{1})$
IIId	$E_6 \times \mathrm{SU}(3) \times E_7 \times \mathrm{U}(1)$ $\frac{1}{3}(1^2, 2, 0^5) \times \frac{1}{3}(1^2, 0^6)$	$(\mathbf{27},\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{1},\mathbf{56})_1$ $+2(\mathbf{1})_0 + (\mathbf{1})_2$	$9(\mathbf{27},\mathbf{1},\mathbf{1})_{\frac{2}{3}} + 9(\mathbf{1},\mathbf{3},\mathbf{1})_{\frac{4}{3}}$ $+18(\mathbf{1},\mathbf{3},\mathbf{1})_{-\frac{2}{3}}$
IIIe	$\mathrm{SU}(9) \times E_6 \times \mathrm{SU}(3)$ $\frac{1}{3}(1^4, 2, 0^3) \times \frac{1}{3}(1^2, 2, 0^5)$	$(\mathbf{84},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{27},\mathbf{3}) + 2(\mathbf{1})$	$9(\mathbf{9},\mathbf{1},\mathbf{3})$

Table 9: Perturbative $E_8 \times E_8$ heterotic orbifold spectra on $T^4/\mathbb{Z}_N$ for $N = 2, 3$.

The actual $E_8 \times E_8$ shift vectors come from a combination of two E_8 shift subvectors. The possible combinations, allowed by modular invariance, have been classified in [18] and are listed in table 11, as well as the gauge symmetry breaking and matter content which is computed here for the first time.

Twisted matter Neglecting the gauge enhancement $\mathrm{SO}(16) \to E_8$, the computation of the twisted matter can be done exactly along the lines explained in section 2.1 for the

Shift	SO(16) breaking	Untwisted matter	E_8 enhancement	Untwisted matter
$\frac{1}{4}\left(0^8\right)$	SO(16)	-	E_8	-
$\frac{1}{4}\left(4,0^7\right)$	SO(16)	-	SO(16)	-
$\frac{1}{4}\left(2,0^7\right)$	SO(14)$\times$U(1)	$(\mathbf{64})_{\frac{1}{2}}$	SO(14)$\times$U(1)	$(\mathbf{64})_{\frac{1}{2}}$
$\frac{1}{4}\left(1^2,0^6\right)$	SO(12)$\times$SU(2)$\times$U(1)	$(\mathbf{12},\mathbf{2})_1+(\mathbf{32}_+,\mathbf{1})_1$	$E_7\times$U(1)	$(\mathbf{56},\mathbf{1})_1$
$\frac{1}{4}\left(1,3,0^6\right)$	SO(12)$\times$SU(2)$\times$U(1)	$(\mathbf{12},\mathbf{2})_1+(\mathbf{32}_+,\mathbf{1})_1$	SO(12)$\times$SU(2)$\times$U(1)	$(\mathbf{12},\mathbf{2})_1+(\mathbf{32}_+,\mathbf{1})_1$
$\frac{1}{4}\left(2^2,0^6\right)$	SO(12)$\times$SO(4)	-	$E_7\times$SU(2)	-
$\frac{1}{4}\left(1^2,2,0^5\right)$	SO(10)$\times$SU(2)$\times$U(1)^2	$(\mathbf{10},\mathbf{2})_{-2,-1}+(\mathbf{16},\mathbf{2})_{1,-1}$ $+(\mathbf{1},\mathbf{2})_{4,-1}+(\mathbf{1},\mathbf{2})_{0,3}$	$E_6\times$SU(2)$\times$U(1)	$(\mathbf{27},\mathbf{2})_{-1}+(\mathbf{1},\mathbf{2})_3$
$\frac{1}{4}\left(2^3,0^5\right)$	SO(10)$\times$SO(6)	$(\mathbf{16},\mathbf{4})$	SO(10)$\times$SO(6)	$(\mathbf{16},\mathbf{4})$
$\frac{1}{4}\left(1^5,3,0^2\right)$	SU(6)$\times$SU(2)$^2\times$U(1)	$(\mathbf{6},\mathbf{2},\mathbf{2})_{-1}+(\mathbf{15},\mathbf{1},\mathbf{2})_1$ $+(\mathbf{1},\mathbf{1},\mathbf{2})_{-3}$	SU(8)$\times$SU(2)	$(\mathbf{28},\mathbf{2})$
$\frac{1}{4}\left(1^7,-1\right)$	SU(8)$\times$U(1)	$(\mathbf{8})_{-3}+(\mathbf{56})_{-1}$	SU(8)$\times$U(1)	$(\mathbf{8})_{-3}+(\mathbf{56})_{-1}$

Table 10: Gauge groups and untwisted matter for the ten inequivalent $\mathbb{Z}_4$ shift vectors of E_8. In the second and third column, the gauge groups and untwisted matter obtained along the same lines as for the SO(32) cases are given, and in the last two columns the enhancements due to the breaking of the **128** gauge boson are listed.

SO(32) case. We are then able to write tables similar to 2 and 3 (6 in the g^2-twisted case) that summarize in an abstract way the whole set of models. Of course, in this approach the twisted matter is arranged in representations of the subgroup of SO(16) $\times$ SO(16) that survives the orbifold projection. If the latter is enhanced, we have to recombine the twisted states into representations of the enhanced gauge symmetry as well.

The complete massless spectra are listed in table 11.

2.3 Examples of $T^4/\mathbb{Z}_6$ orbifold vacua

The 60 inequivalent shift vectors for $E_8\times E_8$ orbifolds on $T^4/\mathbb{Z}_6$ have been classified in [18], for the gauge group SO(32) roughly the same number of different spectra is expected. The computation of gauge symmetry breaking and untwisted spectrum is exactly as described in sections 2.1, with the caveat that, in the $E_8\times E_8$ case, a gauge enhancement is generically expected, as shown previously in the $\mathbb{Z}_4$ case in table 10. About the twisted spectrum, it contains three sectors, that can be computed following the approach used in the $\mathbb{Z}_4$ case. In detail, the only new sector is the g-twisted one, since the g^2-twisted sector corresponds to a $\mathbb{Z}_6$ projection of the g-twisted sector of the $\mathbb{Z}_3$ model, and the g^3-sector in a projection of the twisted sector of the $\mathbb{Z}_2$ model. We do not attempt a complete classification of these models, but rather give a few examples, summarized in table 12.

2.4 Anomaly polynomials for the orbifold models

The anomaly polynomial is computed for each spectrum separately along the lines described in detail in [21] up to the overall normalization change of the anomaly polynomial by a factor of (-16) which leads to a prefactor 1 of the gravitational part in the expansion for perturbative vacua $I_8 = (\mathrm{tr}R^2)^2 + \cdots$. This normalization agrees with the one used in [16]

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Gauge group & Shift		Matter
IVa	U	$(\mathbf{56};\mathbf{1})_1 + 2\,(\mathbf{1};\mathbf{1})_0$
$E_7\times \mathrm{U}(1)\times E_8$	T	$4\,(\mathbf{56};\mathbf{1})_{-\frac{1}{2}} + 8\,(\mathbf{1};\mathbf{1})_{\frac{3}{2}} + 24\,(\mathbf{1};\mathbf{1})_{\frac{1}{2}}$
$\frac{1}{4}(1^2,0^6;0^8)$	T^2	$5(\mathbf{56};\mathbf{1})_0 + 32\,(\mathbf{1};\mathbf{1})_1$
IVb	U	$(\mathbf{56};\mathbf{1},\mathbf{1})_1 + 2\,(\mathbf{1};\mathbf{1},\mathbf{1})_0$
$E_7\times \mathrm{U}(1)\times E_7\times \mathrm{SU}(2)$	T	$12\,(\mathbf{1};\mathbf{1},\mathbf{2})_{\frac{1}{2}} + 4\,(\mathbf{1};\mathbf{1},\mathbf{2})_{-\frac{3}{2}} + 4\,(\mathbf{1};\mathbf{56},\mathbf{1})_{\frac{1}{2}}$
$\frac{1}{4}(1^2,0^6;2^2,0^6)$	T^2	$32\,(\mathbf{1};\mathbf{1},\mathbf{1})_1 + 3\,(\mathbf{56};\mathbf{1},\mathbf{1})_0$
IVc	U	$(\mathbf{56};\mathbf{1})_1 + 2\,(\mathbf{1};\mathbf{1})_0$
$E_7\times \mathrm{U}(1)\times \mathrm{SO}(16)$	T	$8\,(\mathbf{1};\mathbf{16})_{\frac{1}{2}}$
$\frac{1}{4}(1^2,0^6;4,0^7)$	T^2	$32\,(\mathbf{1};\mathbf{1})_1 + 5\,(\mathbf{56};\mathbf{1})_0$
IVd	U	$(\mathbf{56};\mathbf{1})_{1,0} + (\mathbf{1},\mathbf{8})_{0,-3} + (\mathbf{1},\mathbf{56})_{0,-1} + 2\,(\mathbf{1},\mathbf{1})_{0,0}$
$E_7\times \mathrm{U}(1)\times \mathrm{SU}(8)\times \mathrm{U}(1)$	T	$12\,(\mathbf{1};\mathbf{1})_{\frac{1}{2},2} + 4\,(\mathbf{1};\mathbf{1})_{-\frac{3}{2},2} + 4\,(\mathbf{1};\overline{\mathbf{28}})_{\frac{1}{2},0} + 8\,(\mathbf{1};\mathbf{8})_{\frac{1}{2},-1}$
$\frac{1}{4}(1^2,0^6;1^7,-1)$	T^2	$10\,(\mathbf{1};\mathbf{8})_{1,1} + 6\,(\mathbf{1};\mathbf{8})_{-1,1}$
IVe	U	$(\mathbf{27},\mathbf{2};\mathbf{1})_{-1,0} + (\mathbf{1},\mathbf{2};\mathbf{1})_{3,0} + (\mathbf{1},\mathbf{1};\mathbf{64})_{0,\frac{1}{2}} + 2\,(\mathbf{1},\mathbf{1};\mathbf{1})_{0,0}$
$E_6\times \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{SO}(14)\times \mathrm{U}(1)$	T	$12(\mathbf{1},\mathbf{1};\mathbf{1})_{\frac{3}{2},\frac{1}{2}} + 8(\mathbf{1},\mathbf{2};\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} +$ $4(\mathbf{27},\mathbf{1};\mathbf{1})_{\frac{1}{2},-\frac{1}{2}} + 4(\mathbf{1},\mathbf{1};\mathbf{14})_{\frac{3}{2},-\frac{1}{2}}$
$\frac{1}{4}(2,1^2,0^5;2,0^7)$	T^2	$3\,(\mathbf{1},\mathbf{2};\mathbf{14})_{0,0} + 10\,(\mathbf{1},\mathbf{2};\mathbf{1})_{0,1}$
IVf	U	$(\mathbf{27},\mathbf{2};\mathbf{1},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})_3 + (\mathbf{1},\mathbf{1};\mathbf{16},\mathbf{4})_0 + 2\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_0$
$E_6\times \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{SO}(10)\times \mathrm{SO}(6)$	T	$8\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{4})_{\frac{3}{2}} + 4\,(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{4})_{-\frac{3}{2}} + 4\,(\mathbf{1},\mathbf{1};\overline{\mathbf{16}},\mathbf{1})_{\frac{3}{2}}$
$\frac{1}{4}(2,1^2,0^5;2^3,0^5)$	T^2	$5\,(\mathbf{1},\mathbf{2};\mathbf{10},\mathbf{1})_0 + 3\,(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{6})_0$
IVg	U	$(\mathbf{12},\mathbf{2};\mathbf{1})_1 + (\mathbf{32}_+,\mathbf{1};\mathbf{1})_1 + 2\,(\mathbf{1},\mathbf{1};\mathbf{1})_0$
$\mathrm{SO}(12)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times E_8$	T	$12\,(\mathbf{1},\mathbf{2};\mathbf{1})_{-\frac{1}{2}} + 8\,(\mathbf{12},\mathbf{1};\mathbf{1})_{\frac{1}{2}} + 4\,(\mathbf{1},\mathbf{2};\mathbf{1})_{\frac{3}{2}} + 4\,(\mathbf{32}_-,\mathbf{1};\mathbf{1})_{-\frac{1}{2}}$
$\frac{1}{4}(3,1,0^6;0^8)$	T^2	$32\,(\mathbf{1},\mathbf{1};\mathbf{1})_1 + 5\,(\mathbf{12},\mathbf{2};\mathbf{1})_0 + 3\,(\mathbf{32}_+,\mathbf{1};\mathbf{1})_0$
IVh	U	$(\mathbf{12},\mathbf{2};\mathbf{1},\mathbf{1})_1 + (\mathbf{32}_+,\mathbf{1};\mathbf{1},\mathbf{1})_1 + 2\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_0$
$\mathrm{SO}(12)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times E_7\times \mathrm{SU}(2)$	T	$8\,(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})_{-\frac{1}{2}} + 4\,(\mathbf{12},\mathbf{1};\mathbf{1},\mathbf{2})_{\frac{1}{2}}$
$\frac{1}{4}(3,1,0^6;2^2,0^6)$	T^2	$32\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_1 + 3\,(\mathbf{12},\mathbf{2};\mathbf{1},\mathbf{1})_0 + 5\,(\mathbf{32}_+,\mathbf{1};\mathbf{1},\mathbf{1})_0$
IVi	U	$(\mathbf{12},\mathbf{2};\mathbf{1})_1 + (\mathbf{32}_+,\mathbf{1};\mathbf{1})_1 + 2\,(\mathbf{1},\mathbf{1};\mathbf{1})_0$
$\mathrm{SO}(12)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{SO}(16)$	T	$4\,(\mathbf{1},\mathbf{2};\mathbf{16})_{-\frac{1}{2}} +$
$\frac{1}{4}(3,1,0^6;4,0^7)$	T^2	$32\,(\mathbf{1},\mathbf{1};\mathbf{1})_1 + 5\,(\mathbf{12},\mathbf{2};\mathbf{1})_0 + 3\,(\mathbf{32}_+,\mathbf{1};\mathbf{1})_0$
IVj	U	$(\mathbf{28},\mathbf{2};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{16},\mathbf{4}) + 2\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$
$\mathrm{SU}(8)\times \mathrm{SU}(2)\times \mathrm{SO}(10)\times \mathrm{SO}(6)$	T	$4\,(\mathbf{8};\mathbf{1};\mathbf{1},\mathbf{4})$
$\frac{1}{4}(3,1^5,0^2;2^3,0^5)$	T^2	$5\,(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{6}) + 3\,(\mathbf{1},\mathbf{2};\mathbf{10},\mathbf{1})$
IVk	U	$(\mathbf{28},\mathbf{2};\mathbf{1})_0 + (\mathbf{1},\mathbf{1};\mathbf{64})_{\frac{1}{2}} + 2\,(\mathbf{1},\mathbf{1};\mathbf{1})_0$
$\mathrm{SU}(8)\times \mathrm{SU}(2)\times \mathrm{SO}(14)\times \mathrm{U}(1)$	T	$8\,(\overline{\mathbf{8}},\mathbf{1};\mathbf{1})_{\frac{1}{2}} + 4\,(\mathbf{8},\mathbf{2};\mathbf{1})_{\frac{1}{2}}$
$\frac{1}{4}(3,1^5,0^2;2,0^7)$	T^2	$5\,(\mathbf{1},\mathbf{2};\mathbf{14})_0 + 6\,(\mathbf{1},\mathbf{2};\mathbf{1})_1$
IVl	U	$(\mathbf{8};\mathbf{1},\mathbf{1})_{-3,0} + (\mathbf{56};\mathbf{1},\mathbf{1})_{-1,0} + (\mathbf{1};\mathbf{12},\mathbf{2})_{0,1} +$ $(\mathbf{1};\mathbf{32}_+,\mathbf{1})_{0,1} + 2(\mathbf{1};\mathbf{1},\mathbf{1})_{0,0}$
$\mathrm{SU}(8)\times \mathrm{U}(1)\times \mathrm{SO}(12)\times \mathrm{SU}(2)\times \mathrm{U}(1)$	T	$8\,(\mathbf{1};\mathbf{1},\mathbf{2})_{2,-\frac{1}{2}} + 4\,(\mathbf{1};\mathbf{12},\mathbf{1})_{2,\frac{1}{2}} + 4\,(\mathbf{8};\mathbf{1},\mathbf{2})_{-1,-\frac{1}{2}} +$
$\frac{1}{4}(1^7,-1;3,1,0^6)$	T^2	$10\,(\mathbf{8};\mathbf{1},\mathbf{1})_{1,1} + 6\,(\mathbf{8};\mathbf{1},\mathbf{1})_{1,-1}$

Table 11: Perturbative $E_8 \times E_8$ spectra on $T^4/\mathbb{Z}_4$. The first column containing shift vectors and gauge groups goes back to [18], the complete matter content is computed here for the first time. The standard embedding IVa is (up to charge normalization) taken from [21]. The spectrum of model IVj has been listed in [23] with the shift vector $\frac{1}{4}(-7,1^7;-3,1^3,0^4)$, and IVf appears in [24] with $\frac{1}{4}(1^2,-2,0^5;1^3,-3,0^4)$.

Shift & gauge group		Matter
6a	U	$(\mathbf{28},\mathbf{2})_1 + 2\ (\mathbf{1},\mathbf{1})_0$
$\mathrm{SO}(28)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	T	$8\ (\mathbf{1},\mathbf{1})_{\frac{1}{3}} + 2\ (\mathbf{1},\mathbf{1})_{-\frac{5}{3}} + (\mathbf{28},\mathbf{2})_{-\frac{2}{3}}$
$\frac{1}{6}\left(0^{14},1^2\right)$	T^2	$22\ (\mathbf{1},\mathbf{1})_{\frac{2}{3}} + 10\ (\mathbf{1},\mathbf{1})_{-\frac{4}{3}} + 5\ (\mathbf{28},\mathbf{2})_{-\frac{1}{3}}$
	T^3	$22\ (\mathbf{1},\mathbf{1})_1 + 3\ (\mathbf{28},\mathbf{2})_0$
6b	U	$(\mathbf{12},\mathbf{5},\mathbf{1})_1 + 2\ (\mathbf{1},\mathbf{1},\mathbf{1})_0$
$\mathrm{SO}(12)\times\mathrm{SU}(5)\times\mathrm{SO}(10)\times\mathrm{U}(1)$	T	$2\ (\mathbf{1},\mathbf{1},\mathbf{16})_{\frac{5}{6}}$
$\frac{1}{6}\left(0^6,1^5,3^5\right)$	T^2	$\begin{cases} 10(\mathbf{1},\mathbf{5},\mathbf{1})_{-\frac{2}{3}} + 4(\mathbf{1},\mathbf{1},\mathbf{10})_{\frac{5}{3}} + \\ 5(\mathbf{12},\mathbf{1},\mathbf{1})_{\frac{5}{3}} + 4(\mathbf{1},\mathbf{10},\mathbf{1})_{-\frac{4}{3}} \end{cases}$
	T^3	$3\ (\mathbf{32}_+,\mathbf{1},\mathbf{1})_0$
VIa	U	$(\mathbf{56},\mathbf{1})_1 + 2\ (\mathbf{1},\mathbf{1})_0$
$E_7\times\mathrm{U}(1)\times E_8$	T	$8\ (\mathbf{1},\mathbf{1})_{\frac{1}{3}} + 2\ (\mathbf{1},\mathbf{1})_{-\frac{5}{3}} + (\mathbf{56},\mathbf{1})_{-\frac{2}{3}}$
$\frac{1}{6}\left(1^2,0^6;0^8\right)$	T^2	$22\ (\mathbf{1},\mathbf{1})_{\frac{2}{3}} + 10\ (\mathbf{1},\mathbf{1})_{-\frac{4}{3}} + 5\ (\mathbf{56},\mathbf{1})_{-\frac{1}{3}}$
	T^3	$22\ (\mathbf{1},\mathbf{1})_1 + 3\ (\mathbf{56},\mathbf{1})_0$
VIb	U	$\begin{cases} (\mathbf{12},\mathbf{1})_{1,0,0} + (\mathbf{32}_-,\mathbf{1})_{\frac{1}{2},-\frac{1}{2},0} + \\ (\mathbf{1},\mathbf{64})_{0,0,\frac{1}{2}} + 2(\mathbf{1},\mathbf{1})_{0,0,0} \end{cases}$
$\mathrm{SO}(12)\times\mathrm{SO}(14)\times\mathrm{U}(1)^3$	T	$\begin{cases} 5(\mathbf{1},\mathbf{1})_{\frac{1}{6},\frac{1}{2},\frac{1}{3}} + 3(\mathbf{1},\mathbf{1})_{-\frac{5}{6},-\frac{1}{2},\frac{1}{3}} + \\ (\mathbf{1},\mathbf{1})_{\frac{7}{6},-\frac{1}{2},\frac{1}{3}} + 2(\mathbf{12},\mathbf{1})_{\frac{1}{6},-\frac{1}{2},\frac{1}{3}} + \\ (\mathbf{1},\mathbf{14})_{\frac{1}{6},\frac{1}{2},-\frac{2}{3}} + (\mathbf{32}_+,\mathbf{1})_{-\frac{1}{3},0,\frac{1}{3}} + \end{cases}$
$\frac{1}{6}\left(0^6,1,3;0^7,2\right)$	T^2	$\begin{cases} 10(\mathbf{1},\mathbf{1})_{-\frac{2}{3},0,\frac{2}{3}} + 5(\mathbf{12},\mathbf{1})_{\frac{1}{3},0,\frac{2}{3}} + \\ 4(\mathbf{1},\mathbf{14})_{-\frac{2}{3},0,-\frac{1}{3}} + 4(\mathbf{1},\mathbf{1})_{\frac{1}{3},\pm 1,\frac{2}{3}} \end{cases}$
	T^3	$\begin{cases} 5(\mathbf{1},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},1} + 6(\mathbf{1},\mathbf{1})_{\frac{1}{2},-\frac{1}{2},-1} \\ +5(\mathbf{1},\mathbf{14})_{\frac{1}{2},-\frac{1}{2},0} \end{cases}$
VIc	U	$(\mathbf{6},\mathbf{3},\mathbf{2};\mathbf{1}) + 2(\mathbf{1})$
$\mathrm{SU}(6)\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(9)$	T	$(\mathbf{6},\mathbf{1},\mathbf{1};\overline{\mathbf{9}})$
$\frac{1}{6}(-5,1^5,0^2;-5,1^7)$	T^2	$4(\mathbf{1},\mathbf{3},\mathbf{1};\mathbf{9})$
	T^3	$3(\mathbf{20},\mathbf{1},\mathbf{1};\mathbf{1}) + 5(\mathbf{6},\mathbf{3},\mathbf{1};\mathbf{1}) + 10(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})$
VId	U	$\begin{cases} (\mathbf{6},\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{1})_0 + 2(\mathbf{1})_0 + \\ (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{10},\mathbf{4})_1 + (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{4})_5 \end{cases}$
$\mathrm{SU}(6)\times\mathrm{SU}(3)\times\mathrm{SU}(2)$ $\times\mathrm{SU}(5)\times\mathrm{SU}(4)\times\mathrm{U}(1)$	T	$2(\mathbf{6},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})_{\frac{10}{3}} + (\mathbf{6},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{4}})_{-\frac{5}{3}} + (\mathbf{1},\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{1})_{\frac{10}{3}}$
$\frac{1}{6}(-5,1^5,0^2;-4,1^4,0^3)$	T^2	$4(\mathbf{1},\mathbf{3},\mathbf{1};\mathbf{5},\mathbf{1})_{\frac{4}{3}} + 5(\mathbf{1},\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{4})_{-\frac{5}{3}}$
	T^3	$5(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{5},\mathbf{1})_{-2} + 3(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{6})_0$

Table 12: Gauge group and matter for two SO(32) and four $E_8\times E_8$ heterotic string models on $T^4/\mathbb{Z}_6$. The standard embedding VIa can be found in [21] (up to charge normalization), model VIc is taken from [23] and VId from [24].

for the smooth $K3$ embeddings with which the orbifold point anomaly polynomials will be compared in section 3.

It turns out that there occurs a very general form, namely for SO(32) heterotic orbifold

#	$2M$	α	N	β	γ	$\tilde{\gamma}$
2c	-	-	16	0	-512	-32
3,4,6a	28	2	2	-44	-24	-4
3b	20	2	5	-8	-140	-10
3c	16	2	8	-8	-64	-16
3d	$10^\star$	$-\frac{5}{2}$	11	1	-231	-22
4a'	28	2	2	-28	-24	-4
4e	$12^\star$	-1	10	-2	-20	-20
4e'	$12^\star$	-3	10	2	-80	-20

Table 13: Coefficients in the SO(32) anomaly polynomials for models with up to one gauge group of each kind, $\mathrm{SO}(2M) \times \mathrm{SU}(N) \times \mathrm{U}(1)$. A star at the value of $2M$ indicates that the massless spectrum contains spinor representations of $\mathrm{SO}(2M)$. For all models, we have $\tilde{\gamma} = -2N$, which coincides with the expected value for the $\mathrm{U}(1)$ to be the trace part of an $\mathrm{U}(N)$ gauge factor. However, γ does not match with the smooth models discussed in section 3.2, and from the explicit spectra in tables 1, 7, 8 and 12, one sees that instead of being the trace part of some larger group, the $\mathrm{U}(1)$ charges are associated to the twist sectors of the orbifold.

models we obtain

$$I_8^{\mathrm{SO}(32)} = \left(\mathrm{tr}R^2 + \sum_i \alpha_i \mathrm{tr}_{\mathrm{SO}(2M_i)}F^2 + \sum_j \beta_j \mathrm{tr}_{\mathrm{SU}(N_j)}F^2 + \sum_k \gamma_k F^2_{\mathrm{U}(1)_k} + \sum_{i<j} \delta_{ij} F_{\mathrm{U}(1)_i} F_{\mathrm{U}(1)_j}\right) \times \\ \times \left(\mathrm{tr}R^2 - \sum_i \mathrm{tr}_{\mathrm{SO}(2M_i)}F^2 - 2\sum_j \mathrm{tr}_{\mathrm{SU}(N_j)}F^2 + \sum_k \tilde{\gamma}_k F^2_{\mathrm{U}(1)_k}\right) \tag{2.25}$$

with the coefficients $\alpha, \beta, \gamma, \delta$ and $\tilde{\gamma}$ listed in tables 13, 14 and 15.

Also for the $E_8 \times E_8$ case, the anomaly polynomial can be cast into a generic form: label non-Abelian groups of rank r_i descending from $E_8^{(i)}$ $(i = 1, 2)$ by G_{r_i}. Then the anomaly polynomial for $E_8 \times E_8$ orbifold compactifications to six dimensions has the general form

$$I_8^{E_8 \times E_8} = \left(\mathrm{tr}R^2 - \sum_{i=1}^{2}\sum_x a^x_{r_i} \mathrm{tr}_{G^x_{r_i}} F^2 - \sum_y b_y F^2_{\mathrm{U}(1)_y} + \sum_{y<z} c_{yz} F_{\mathrm{U}(1)_y} F_{\mathrm{U}(1)_z}\right) \times \\ \times \left(\mathrm{tr}R^2 - \sum_{i=1}^{2}\sum_x \tilde{a}^x_{r_i} \mathrm{tr}_{G^x_{r_i}} F^2 - \sum_y \tilde{b}_y F^2_{\mathrm{U}(1)_y}\right). \tag{2.26}$$

The coefficients $a^x_{r_i}, b_y, c_{yz}, \tilde{b}_y$ depend on the combination of two shift vectors and are listed in table 16, whereas the coefficients $\tilde{a}^x_r$ are universal for fixed gauge groups,

G_r	E_8	E_7	E_6	$\mathrm{SO}(2M)$	$\mathrm{SU}(N)$
$\tilde{a}_r$	1	$\frac{1}{6}$	$\frac{1}{3}$	1	2

(2.27)

and coincide for $\mathrm{SO}(2M)$ and $\mathrm{SU}(N)$ with those of the SO(32) heterotic orbifolds. It turns out that the instanton number k inside an E_8 gauge factor (with $k_1 + k_2 = 24$) listed in

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#	$2M_1$	α_1	$2M_2$	α_2	N_1	β_1	N_2	β_2	N_3	β_3	γ	$\tilde{\gamma}$
2a	28	2	-	-	2	-44	2	-12	-	-	-	-
2b	20	2	$12^\star$	-6	-	-	-	-	-	-	-	-
3e	-	-	-	-	14	1	2	-23	2	-5	-182	-28
4b	24	2	-	-	2	-20	2	-4	2	-20	-24	-4
4c	20	2	$8^\star$	-2	2	-28	-	-	-	-	-16	-4
4d	16	2	$12^\star$	-2	2	-20	-	-	-	-	-16	-4
4f	$8^\star$	-3	-	-	10	2	2	-6	2	-14	-80	-20
4h	14	2	-	-	$6^\star$	-8	4	-4	-	-	-48	-12
4i	$10^\star$	0	$10^\star$	0	$6^\star$	-4	-	-	-	-	-96	-12
6b	$12^\star$	-1	$10^\star$	1	5	-2	-	-	-	-	$-\frac{590}{9}$	-10

Table 14: Coefficients for models with several SO($2M$) and SU(N) factors. For all models with an U(1) gauge factor, we have $\tilde{\gamma} = -2N_1$. A star indicates that the massless spectrum contains representation of the gauge groups that cannot be reproduced in the smooth case (spinorial representation of SO($2M$), third rank totally antisymmetric representation of SU(N) etc.).

#	$2M$	α	N_1	β_1	N_2	β_2	γ_1	$\tilde{\gamma}_1$	γ_2	$\tilde{\gamma}_2$	γ_3	$\tilde{\gamma}_3$	δ_{12}	δ_{13}	δ_{23}
4g	18	2	$6^\star$	-8	-	-	-84	-12	-8	-2	-	-	24	-	-
4j	-	-	14	2	-	-	-16	-2	-168	-28	-16	-2	28	-16	28
4k	-	-	15	1	-	-	$-\frac{61}{8}$	-2	$-\frac{2805}{8}$	-30	-	-	$\frac{15}{4}$	-	-
4l	-	-	11	1	5	-3	$-\frac{595}{8}$	-10	$-\frac{1243}{8}$	-22	-	-	$\frac{385}{4}$	-	-
4m	-	-	9	1	7	-3	$-\frac{1143}{8}$	-18	$-\frac{511}{8}$	-14	-	-	$\frac{189}{4}$	-	-
4n	-	-	13	1	3	-11	$-\frac{1417}{8}$	-26	$-\frac{129}{8}$	-6	-	-	$\frac{195}{4}$	-	-

Table 15: Coefficients for models with several U(1) factors. For 4g, 4j, 4m, 4n we have $\tilde{\gamma}_1 = -2N_1$, for 4k, 4l $\tilde{\gamma}_2 = -2N_1$. Furthermore, for 4l we have $\tilde{\gamma}_1 = -2N_2$, and for 4m, 4n $\tilde{\gamma}_2 = -2N_2$.

table 16 is for all $T^4/\mathbb{Z}_N$ models with $N = 2, 3, 4$ related to the coefficients of the largest non-Abelian gauge factor by

$$k = 12 + 2\,\frac{a_r^1}{\tilde{a}_r^1}. \tag{2.28}$$

Assuming that this relation holds also for $N = 6$, the coefficient a_r^1 of the largest gauge factor inside each E_8 can be computed for all models from the instanton numbers given in [18] without having to compute the matter spectrum.

For both gauge groups SO(32) and $E_8 \times E_8$, the anomaly polynomials (2.25) and (2.26) factorize completely into 4×4 forms. This is in contrast to smooth compactifications where a sum of two factorized expressions $4\times4+6\times2$ occurs, and the second part signals that U(1)

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gauge factors become massive via the Green-Schwarz mechanism. It is therefore natural to deduce that the U(1) gauge groups at the orbifold point remain massless, however, it should be stressed that the absence of a 6×2 factorized part could in principle also be due to the absence of the six-form part in the presence of a mass term of the U(1) gauge fields providing the two-form part.

Apart from the discrepancy of the Abelian gauge factors, we will see a very similar pattern for the anomaly polynomials in the smooth case in section 3, and the matching of coefficients will be a guiding principle for obtaining the correct second Chern characters (instanton numbers) of the bundles.

3. The heterotic string on $K3$ with line bundles

In this section, the six dimensional heterotic orbifold spectra constructed in section 2 are compared with smooth compactifications on $K3$ with line bundles. After a general discussion of consistency conditions in section 3.1, six dimensional model building on $K3$ of the SO(32) heterotic string is elaborated on in section 3.2 and for the $E_8 \times E_8$ heterotic string in section 3.3. Some explicit matchings of spectra with the orbifold cases are presented for both classes, and it is argued that for other cases there exist obstructions to find smooth matchings with the simple ansatz of embedding just one line bundle presented here. We infer that orbifold models with several twist sectors require multiple line bundles.

3.1 6D spectra, supersymmetry and tadpole cancellation

We review briefly the basic model building features for both SO(32) and $E_8 \times E_8$ heterotic string compactifications on $K3$ discussed in detail in [16]:

- Each hyper multiplet transforming in some representation $\mathbf{R}$ is associated to a bundle V on $K3$. (Minus) the number of hyper multiplets is given by the Riemann-Roch-Hirzebruch theorem,

$$\chi(V)_{K3} \equiv \int_{K3} \mathrm{ch}(V)\, \mathrm{Td}(K3) = \mathrm{ch}_2(V) + 2r, \tag{3.1}$$

where $\mathrm{ch}(V) = r + c_1(V) + \mathrm{ch}_2(V) + \cdots$ is the total Chern character of the bundle V, r its rank and $\mathrm{Td}(K3) = 1 + \frac{1}{12}c_2(K3) + \cdots = 1 + 2\,\mathrm{vol}_4 + \cdots$ the Todd class of the tangent bundle of the $K3$ surface.[8] Since the sign of the supersymmetric index (3.1) corresponds to the chirality of the fermions in a multiplet, in this convention, a *negative* index counts the number of *hyper* multiplets, while a *positive* index implies the existence of *vector* multiplets in the representation $\mathbf{R}$.

  The assignment of representations and bundles will be discussed in section 3.2 for SO(32) embeddings and 3.3 for $E_8 \times E_8$ cases.

- Associated to the bundle V is a background field strength $\overline{F}$ on $K3$. A supersymmetric background requires

[8] Whenever $\mathrm{ch}_2(V)$ appears in an index, integration over $K3$ is understood since any four form is proportional to the normalized volume form, $\int_{K3} \mathrm{vol}_4 = 1$.

	G_r^1	a_r^1	$\mathrm{ch}_2(L)$	G_r^2	a_r^2	b_y^1	$\tilde{b}_y^1$	c_{12}	k
(0^8) II-VIa, IIIc, IVg	E_8	-6	0	-	-	-	-	-	0
III-VIa $\frac{1}{N}(1^2,0^6)$ IIId, IVd $N\neq 2$ IVb IVc	E_7	1 $-\frac{1}{2}$ 0 $\frac{1}{3}$	-12 -3 -6 -8	-	-	24 36 24 16	4	-	24 6 12 16
$\frac{1}{2}(1^2,0^6)$ IIa IIb IVb IVh	E_7	1 $-\frac{1}{3}$ 0 $-\frac{2}{3}$	-12 -4 -6 -2	SU(2)	12 $28^\star$ 0 $8^\star$	-	-	-	24 8 12 4
$\frac{1}{3}(1^2,2,0^5)$ IIId IIIe	E_6	1 $-\frac{1}{2}$	-3 $-\frac{3}{2}$	SU(3)	6 $15^\star$	-	-	-	18 9
$\frac{1}{4}(1^2,2,0^5)$ IVe IVf	E_6	0 $-\frac{2}{3}$	-2 $-\frac{4}{3}$	SU(2)	$12^\star$ $16^\star$	36 48	12	24 –	12 8
$\frac{1}{2}(1,0^7)$ IIb $(1,0^7)$ IVc,i	SO(16)	$2^\star$ $-2^{(i\star)}$	-4 -2	-	-	-	-	-	16 8
IIIb $\frac{1}{N}(2,0^7)$ IVe $N\neq 2$ IVk VIb	SO(14)	0 $0^\star$ $0^\star$ 0	-3	-	-	16 12 8 $\frac{104}{9}$	2	16 24 – $-\frac{56}{9}, \frac{8}{3}$	12
$\frac{1}{4}(3,1,0^6)$ IVg IVh IVi IVl	SO(12)	6 $4^\star$ 2 -1	$-\frac{12}{5}$ -2 $-\frac{8}{5}$ -1	SU(2)	12 $8^\star$ $20^\star$ $6^\star$	24 16 16 32	4	– – – -32	24 20 16 10
$\frac{1}{6}(3,1,0^6)$ VIb	SO(12)	0	$-\frac{6}{5}$	-	-	$\frac{110}{9}$ 6	2	$\frac{28}{3}$, $\frac{-56}{9}$ $\frac{8}{3}$	12
$\frac{1}{4}(2^3,0^5)$ IVf IVj	SO(10)	$2^\star$ $0^\star$	$-\frac{4}{3}$ -1	SU(4)	$4^\star$ $8^\star$	-	-	-	16 12
$\frac{1}{3}(1^4,2,0^3)$ IIIc IIIe	SU(9)	12 $3^\star$	-3 $-\frac{15}{8}$	-	-	-	-	-	24 15
$\frac{1}{6}(-5,1^7)$ VIc	SU(9)	$-2^\star$	$-\frac{5}{16}$	-	-	-	-	-	10
$\frac{1}{4}(1^7,-1)$ IVd IVl	SU(8)	6 $2^\star$	$-\frac{9}{4}$ $-\frac{7}{4}$	-	-	48 80	16	– -32	18 14
$\frac{1}{4}(3,1^5,0^2)$ IVj IVk	SU(8)	$0^\star$ 0	$-\frac{6}{7}$	SU(2)	$12^\star$ $20^\star$	-	-	-	12
$\frac{1}{6}(-5,1^5,0^2)$ VIc VId	SU(6)	$4^\star$ $-2^\star$	$-\frac{7}{15}$ $-\frac{1}{3}$	SU(3) SU(2)	VIc : $\begin{cases} 10^\star \\ 2 \end{cases}$ VId : $\begin{cases} 6^\star \\ 8^\star \end{cases}$	-	-	-	14 10
$\frac{1}{6}(-4,1^4,0^2)$ VId	SU(5)	$2^\star$	$-\frac{7}{10}$	SU(4)	$4^\star$	$\frac{320}{3}$	40	-	14

Table 16: Systematics on coefficients for E_8. A star at the value of a_r indicates that there exists matter charged under both this gauge group as well as some gauge factor which arises from the other E_8 breaking; k labels the instanton numbers from one E_8 factor given in [17, 18, 23]. The entries $\mathrm{ch}_2(L)$ correspond to smooth line bundle matchings and are explained in section 3.3.

1. $\overline{F}$ is holomorphic, i.e. a (1,1) form and has no contributions from the (2,0) or (0,2) form.
2. $\overline{F}$ is primitive,

$$\int_{K3} J \wedge \mathrm{tr}\overline{F} = 0, \tag{3.2}$$

where J is the Kähler form on $K3$. The $K3$ lattice is of type (3,19), which means that there exist three self dual and 19 anti-selfdual forms. For the Kähler form J to be well defined, it has to lie in the self dual part of the lattice, and (3.2) is satisfied if $\mathrm{tr}\overline{F}$ is zero or $\overline{F}$ lies in the anti-self dual sublattice.

- The supersymmetry conditions are trivially satisfied for $\mathrm{SU}(n)$ bundles which have vanishing first Chern class. For more general $\mathrm{U}(n)$ bundles, the holomorphicity condition freezes two geometric moduli, and the primitivity fixes a third modulus. In order to preserve supersymmetry, the $\mathrm{U}(1)$ factor inside an observable $\mathrm{U}(N)$ must become massive by absorbing a complete neutral hyper multiplet. The fourth scalar d.o.f. inside such a neutral hyper multiplet is given by the dimensional reduction $b_k^{(0)}$ of the ten dimensional antisymmetric tensor B over a two-cycle inside $K3$ labeled by the index k. The tree level couplings

$$S_{\mathrm{mass}} = \sum_{k=0}^{21} \frac{1}{4\pi\ell_s^4} \int_{\mathbb{R}^{1,5}} c_k^{(4)} \wedge \left[\mathrm{tr}(F\overline{F})\right]^{(k)} \tag{3.3}$$

between the four forms $c_k^{(4)}$ dual to the scalars $b_k^{(0)}$ and the $\mathrm{U}(1)$ gauge field generate the mass.

Since $\mathrm{tr}(F\overline{F}) = \sum_i a_i c_1(V_i)\ F_{\mathrm{U}(1)_i}$ holds with the coefficients a_i depending on the specific embedding, in the generic case of different V_i, all $\mathrm{U}(1)$ gauge factors become massive, and linear combinations of the $\mathrm{U}(1)$s remain massless only if the first Chern classes of the respective bundles are linearly dependent.

- The Bianchi identity on the three form field strength $H = dB - \frac{\alpha'}{4}(\omega_{\mathrm{YM}} - \omega_L)$ results in the so called "tadpole cancellation condition" on the background fields,

$$\mathrm{tr}\overline{F}^2 - \mathrm{tr}\overline{R}^2 = 0 \tag{3.4}$$

in cohomology. In this article, we restrict ourselves to *perturbative* vacua only; the generalization to including H5-branes is straight forward as discussed in [16].

The Bianchi identity is replaced in orbifold compactifications by the quadratic level matching condition on the shift vectors (2.8).

- The so called "K-theory constraint" requires

$$c_1(W_{\mathrm{total}}) \in H^2(K3, 2\mathbb{Z}) \tag{3.5}$$

for the total bundle W_{total} to admit spinors. Moreover, any bundle V has $c_1(V) \in H^2(K3, \mathbb{Z})$.

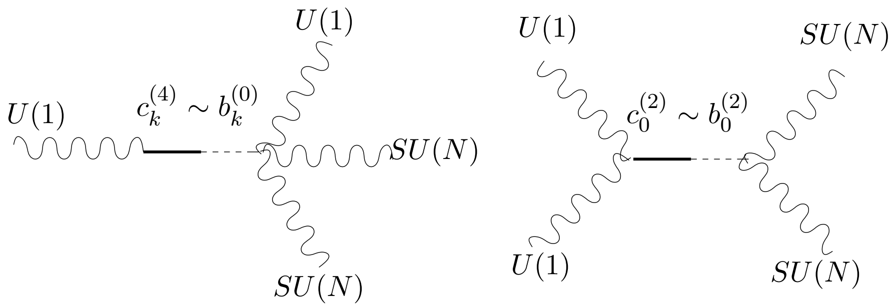

Figure 1: The two possible types of Green-Schwarz counter diagrams. The couplings on the left in the 2×6 factorized diagram generate $\mathrm{U}(1)$ masses.

In the orbifold case, this corresponds to the linear modular invariance constraint on the shift vectors (2.7).

- The Green-Schwarz mechanism consists of two types of counter diagrams: the eight-form anomaly polynomial is a sum of a factorization as 2×6 and another as 4×4 forms,

$$\mathcal{I}_{\text{pert}} = \frac{1}{48(2\pi\ell_s)^4} \int_{K3} \left(\text{tr}(F\overline{F}) \wedge X_{\overline{2}+6} + \frac{1}{2} \left(\text{tr}F^2 - \text{tr}R^2 \right) \wedge X_{\overline{4}+4} \right), \tag{3.6}$$

respectively, as shown in figure 1. The first term in (3.6) corresponds to the exchange of the four forms $c_k^{(4)}$ and their scalar duals $b_k^{(0)}$ and involves always at least one Abelian gauge factor. The last term has the two form $c_0^{(2)}$ and its dual $b_0^{(2)}$ (which is B truncated to six dimensions) as internal propagating fields and contributes to the cancellation of pure and mixed gravitational and non-Abelian anomalies as well as Abelian ones. $X_{\overline{m}+n}$ labels the eight form appearing in the ten dimensional Green Schwarz counter term with n indices along $\mathbb{R}^{1,5}$ and $\overline{m}$ indices on $K3$.

3.2 $\mathrm{U}(n)$ bundles inside $\mathrm{SO}(32)$

The starting point for matching $\mathrm{SO}(32)$ heterotic orbifold vacua with smooth $K3$ compactifications is the group theoretical decomposition $\mathrm{SO}(32) \to \mathrm{SO}(2M) \times \prod_i \mathrm{U}(N_i\, n_i)$ and its adjoint representation,[9]

$$\mathbf{496} \to \begin{pmatrix} (\mathbf{Anti}_{\mathrm{SO}(2M)}) \\ \sum_{j=1}^{K} (\mathbf{Adj}_{\mathrm{U}(N_j)}; \mathbf{Adj}_{\mathrm{U}(n_j)}) \\ \sum_{j=1}^{K} (\mathbf{Anti}_{\mathrm{U}(N_j)}; \mathbf{Sym}_{\mathrm{U}(n_j)}) + (\mathbf{Sym}_{\mathrm{U}(N_j)}; \mathbf{Anti}_{\mathrm{U}(n_j)}) + c.c. \\ \sum_{i<j} (\mathbf{N}_i, \mathbf{N}_j; \mathbf{n}_i, \mathbf{n}_j) + (\mathbf{N}_i, \overline{\mathbf{N}}_j, \mathbf{n}_i, \overline{\mathbf{n}}_j) + c.c. \\ \sum_{j=1}^{K} (\mathbf{2M}, \mathbf{N}_j; \mathbf{n}_j) + c.c. \end{pmatrix}, \tag{3.7}$$

where $M \equiv 16 - \sum_i N_i\, n_i$. Embedding $\mathrm{U}(n_i)$ bundles V_i inside $\mathrm{U}(N_i\, n_i)$ leads to the massless spectrum listed in table 17, from which the anomaly polynomial in the perturbative

[9]For the same kind of decomposition in the context of CY_3-fold compactifications see [13].

reps.	$H = \mathrm{SO}(2M) \times \prod_{i=1}^{K} \mathrm{SU}(N_i) \times \mathrm{U}(1)_i$
$(\mathbf{Adj}_{\mathrm{U}(N_i)})_{0(i)}$	$H^*(K3, V_i \otimes V_i^*)$
$(\mathbf{Sym}_{\mathrm{U}(N_i)})_{2(i)}$	$H^*(K3, \bigwedge^2 V_i)$
$(\mathbf{Anti}_{\mathrm{U}(N_i)})_{2(i)}$	$H^*(K3, \bigotimes_s^2 V_i)$
$(\mathbf{N}_i, \mathbf{N}_j)_{1(i),1(j)}$	$H^*(K3, V_i \otimes V_j)$
$(\mathbf{N}_i, \overline{\mathbf{N}}_j)_{1(i),-1(j)}$	$H^*(K3, V_i \otimes V_j^*)$
$(\mathbf{Adj}_{\mathrm{SO}(2M)})_0$	$H^*(K3, \mathcal{O})$
$(\mathbf{2M}, \mathbf{N}_i)_{1(i)}$	$H^*(K3, V_i)$

Table 17: Perturbative massless spectrum with the structure group taken to be $G = \prod_{i=1}^{K} \mathrm{U}(n_i)$. The net number of hyper multiplets in complex representations is given by $-\chi(W)$ associated to the cohomology class $H^*(K3, W)$ as defined in (3.1). The massless spectrum contains also the supergravity sector, 20 neutral hypers encoding the $K3$ geometry and the universal tensor multiplet.

smooth case is computed [16],

$$\begin{aligned} I_8^{\mathrm{SO}(32)} = & \left(\mathrm{tr} R^2 + 2\, \mathrm{tr}_{\mathrm{SO}(2M)} F^2 + 4 \sum_i (\mathrm{ch}_2(V_i) + n_i) \mathrm{tr}_{\mathrm{U}(N_i)} F^2 \right) \times \\ & \times \left(\mathrm{tr} R^2 - \mathrm{tr}_{\mathrm{SO}(2M)} F^2 - 2 \sum_i n_i \, \mathrm{tr}_{\mathrm{U}(N_i)} F^2 \right) \\ & + \frac{1}{3} \left(\sum_i c_1(V_i) \mathrm{tr}_{\mathrm{U}(N_i)} F \right) \times \left(\sum_j c_1(V_j) \left[\mathrm{tr}_{\mathrm{U}(N_j)} F \, \mathrm{tr} R^2 - 16 \mathrm{tr}_{\mathrm{U}(N_j)} F^3 \right] \right) . \end{aligned}$$

Comparing with the anomaly polynomials at the orbifold point (2.25) leads to the following observations:

- In all orbifold cases where the $\mathrm{SO}(2M_i)$ group occurs only with fundamental and not with spinor representations, the coefficient $\alpha_i = 2$ matches with the smooth case.
- The non-Abelian parts of the anomaly polynomial containing $\mathrm{SU}(N_i)$ gauge factors match when identifying the orbifold and smooth parameters as follows,

$$\begin{aligned} \beta_i &= 4\,(\mathrm{ch}_2(V_i) + n_i), \\ 1 &= n_i . \end{aligned}$$

  The second equation reveals that the background consists of *line bundles* with structure group $\mathrm{U}(1)$ embedded in $\mathrm{U}(N_i)$, and from the first equation the second Chern characters of these line bundles can be determined using the orbifold data.

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- The $\mathrm{SO}(2M_i)$ gauge factors at the orbifold point with spinor representations in the massless spectrum and $\alpha_i \neq 2$ are replaced by $\mathrm{U}(M_i)$ factors on $K3$ with the identification of coefficients

$$\begin{aligned} \mathrm{SO}(2M_i) &\rightarrow \mathrm{U}(M_i) \\ 2\,\alpha_i &= 4\,(\mathrm{ch}_2(V_i) + n_i), \\ 1 &= n_i. \end{aligned}$$

  As for the previous case, in the smooth compactification a line bundle is embedded in $\mathrm{U}(M_i)$ and the second Chern character of this line bundle can be determined from the corresponding coefficient α_i of the anomaly polynomial at the orbifold point.

- The $\mathrm{U}(1)$ charges at the orbifold point cannot be reproduced by the smooth ansatz as one can easily see from the fact that, e.g., in the smooth case all fundamental representations of $\mathrm{SU}(N_i)$ carry charge 1 under $\mathrm{U}(1)_i$ whereas in the orbifold limit their charge is $1 + \frac{n}{m}$ in the m^{th} twisted sector for some integer n (or half-integer for spinorial shifts) and 1 in the untwisted sector. Similarly, antisymmetric representations in the smooth case and untwisted orbifold sector have $\mathrm{U}(1)$ charge 2, in the m^{th} twisted orbifold sector the charges are $2 + \frac{n'}{m}$.

  Furthermore, at the orbifold point, the complete anomaly polynomial factorizes as 4×4 which suggests that the couplings to the (sometimes not even as twisted singlets identifiable) $c_k^{(4)}$ are absent. The Green-Schwarz counter term involves only the supergravity and the universal tensor multiplet,[10] and thus the $\mathrm{U}(1)$ at the orbifold point is expected to remain massless.[11]

The above comparison shows that the *non-Abelian* charges of the orbifold compactifications can be reproduced by embedding line bundles L_i in $\mathrm{U}(N_i)$ gauge factors, and the general massless spectrum simplifies due to the absence of symmetric representations as shown in table 18. This statement is true up to the caveat that in the smooth case, sometimes only smaller subgroups occur, e.g. $\mathrm{SU}(M)$ gauge factors instead of $\mathrm{SO}(2M)$ with spinor representations or $\mathrm{SU}(N-1)$ instead of $\mathrm{SU}(N)$ with third rank antisymmetric representations. The "tadpole cancellation condition" (3.4) for this kind of embeddings is determined via

$$\begin{aligned} \mathrm{tr}\overline{F}^2 &= \frac{1}{30}\mathrm{Tr}\overline{F}^2 = 2\sum_i N_i \overline{F}^2_{\mathrm{U}(1)_i} = 16\pi^2 \sum_i N_i\,\mathrm{ch}_2(L_i), \\ \mathrm{tr}\overline{R}^2 &= 2\,\mathrm{tr}_f^{\mathrm{SU}(2)}\overline{R}^2 = -16\pi^2 c_2(K3), \end{aligned}$$

[10] The anti-self dual combination of $b_0^{(2)}$ and its dual $c_0^{(2)}$ forms part of the supergravity multiplet, whereas the self dual combination is the tensor of the universal tensor multiplet.

[11] In principle the 2×6 term could also vanish due to zero couplings of the $b_k^{(0)}$ in the presence of mass terms. Whereas in the smooth case, the field theoretic couplings are well understood [16], a similar analysis for the orbifold cases has not been performed in detail, and in cases such as the $T^4/\mathbb{Z}_2$ models, there are non-Abelian singlets only in the untwisted sector, all other potential candidates for $b_k^{(0)}$ fields are only identified in the blown-up phase.

reps.	# Hyper	# Vector
$(\mathbf{Adj}_{\mathrm{U}(N_i)})_0$	0	1
$(\mathbf{Adj}_{\mathrm{SO}(2M)})_0$	0	1
$(\mathbf{Anti}_{\mathrm{U}(N_i)})_{2(i)}$	$-\chi(L_i^2)$	0
$(\mathbf{N}_i, \mathbf{N}_j)_{1(i),1(j)}$	$-\chi(L_i \otimes L_j)$	0
$(\mathbf{N}_i, \overline{\mathbf{N}}_j)_{1(i),-1(j)}$	$-\chi(L_i \otimes L_j^{-1})$	0
$(\mathbf{2M}, \mathbf{N}_i)_{1(i)}$	$-\chi(L_i)$	0

Table 18: Matching perturbative orbifold spectra by embedding line bundles L_i in $\prod_i \mathrm{U}(N_i)$.

with $c_2(K3) = 24$ and reads

$$\sum_i N_i \, \mathrm{ch}_2(L_i) = -24. \tag{3.8}$$

Equation (3.8) serves as a good check for the consistency of a model constructed using the coefficients of the anomaly polynomial at the orbifold point to be explained below.

In analogy to the orbifold case, the embedding of a line bundle L into a $\mathrm{U}(N)$ subgroup of SO(32) can be denoted by

$$(\underbrace{L, \ldots, L}_{N\times}, 0, \ldots, 0). \tag{3.9}$$

Also in complete analogy to the orbifold case, the resulting spectrum is *independent of the sign* of the first Chern class of the line bundle,[12] only the K-theory constraint changes. Therefore, all embeddings

$$(\underbrace{L, \ldots, L}_{N_1\times}, \underbrace{L^{-1}, \ldots, L^{-1}}_{N_2\times}, 0, \ldots, 0).$$

are equivalent, and their total first Chern class is

$$c_1(W_{\mathrm{total}}) = (N_1 - N_2)\, c_1(L). \tag{3.10}$$

The generalization to embeddings with several line bundles or different powers L^n is straightforward.

[12]This can be seen explicitly by computing the multiplicities of bifundamental states if L is embedded in $\mathrm{U}(N_1)$ and L^{-1} in $\mathrm{U}(N_2)$. In this case, $Q_{\mathrm{massless}} = N_2 Q_1 + N_1 Q_2$ is the charge of the massless Abelian gauge factor, and $(\mathbf{N}_1, \mathbf{N}_2)_{1,1}$ has multiplicity $\chi(\mathcal{O}) = 2$ which implies the existence of vector multiplets in the bifundamental representation and thereby the gauge enhancement $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2) \times \mathrm{U}(1)_{\mathrm{massless}} \to \mathrm{SU}(N_1 + N_2)$.

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#	$2M$	N_0	$\mathrm{ch}_2(L_0)$	N_1	$\mathrm{ch}_2(L_1)$	N_2	$\mathrm{ch}_2(L_2)$	N_3	$\mathrm{ch}_2(L_3)$
2a	28	-	-	2	-12	(2)	(-4)	-	-
3,4,6a	28	-	-	2	-12	-	-	-	-
2b	20	6	-4	-	-	-	-	-	-
2c	-	-	-	$16^\star$	-1	-	-	-	-
3b	20	-	-	$5^\star$	-3	-	-	-	-
3c	16	-	-	8	-3	-	-	-	-
3d	-	5	$-\frac{9}{4}$	$11^\star$	$-\frac{3}{4}$	-	-	-	-
3e	-	-	-	14	$-\frac{3}{4}$	2	$-\frac{27}{4}$	(2)	$(-\frac{9}{4})$
4a'	28	-	-	2	-8	-	-	-	-
4b	24	-	-	2	-6	(2)	(-2)	2	-6
4c	20	4	-2	2	-8	-	-	-	-
4d	16	6	-2	2	-6	-	-	-	-
4e	-	6	$-\frac{3}{2}$	$10^\star$	$-\frac{3}{2}$	-	-	-	-
4e'	-	6	$-\frac{5}{2}$	10	$-\frac{1}{2}$	-	-	-	-
4f	-	4	$-\frac{5}{2}$	10	$-\frac{1}{2}$	(2)	$(-\frac{5}{2})$	2	$-\frac{9}{2}$
4g	18	-	-	$6^\star$	-3	-	-	-	-
4h	14	-	-	$6^\star$	-3	4	-2	-	-
4i	-	$5_{i=1,2}$	$-1_{i=1,2}$	$6^\star$	-2	-	-	-	-
4j	-	-	-	14	$-\frac{1}{2}$	-	-	-	-
4k	-	-	-	15	$-\frac{3}{4}$	-	-	-	-
4l	-	-	-	11	$-\frac{3}{4}$	5	$-\frac{7}{4}$	-	-
4m	-	-	-	9	$-\frac{3}{4}$	7	$-\frac{7}{4}$	-	-
4n	-	-	-	13	$-\frac{3}{4}$	3	$-\frac{15}{4}$	-	-
6b	-	6 5	$-\frac{3}{2}$ $-\frac{1}{2}$	5	$-\frac{3}{2}$	-	-	-	-

Table 19: Determining second Chern characters from the anomaly polynomial according to equation (3.11). $\mathrm{SU}(N_0)$ corresponds to an $\mathrm{SO}(2N_0)$ gauge group at the orbifolds with $\alpha_i \neq 2$ and spinor representations in the spectrum. In some $T^4/\mathbb{Z}_{2N}$ cases, one of the SU(2) gauge factors is not reproduced by the smooth ansatz. These factors are displayed in parenthesis. Furthermore a star denotes the fact that representations unavailable in the smooth case occur at the orbifold point, e.g. a **20** of SU(6). In the smooth ansatz, an $\mathrm{SU}(N-1)$ gauge factor occurs for these cases instead of the $\mathrm{SU}(N)$ factor listed.

3.2.1 Matching of SO(32) heterotic orbifold and $K3$ spectra

In this section, we use the general form of the spectrum for compactifications with line

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#	N_1	$\mathrm{ch}_2(L_1)$	N_2	$\mathrm{ch}_2(L_2)$	N_3	$\mathrm{ch}_2(L_3)$
2-6a	2	-12	-	-	-	-
2b	6	-4	-	-	-	-
2c	15	-1	1	-9	-	-
3b	4	-3	1	-12	-	-
3c	8	-3	-	-	-	-
3d	10	$-\frac{3}{4}$	5	$-\frac{9}{4}$	1	$-\frac{21}{4}$
3e	14	$-\frac{3}{4}$	2	$-\frac{27}{4}$	-	-
4b	2	-6	2	-6	-	-
4c	2	-8	4	-2	-	-
4d	2	-6	6	-2	-	-
4e	6	$-\frac{3}{2}$	10	$-\frac{3}{2}$	-	-
4e'	6	$-\frac{5}{2}$	9	$-\frac{1}{2}$	1	$-\frac{9}{2}$
4f	4	$-\frac{5}{2}$	10	$-\frac{1}{2}$	2	$-\frac{9}{2}$
4k	15	$-\frac{3}{4}$	1	$-\frac{51}{4}$	-	-

Table 20: Matching some perturbative SO(32) orbifold spectra on $T^4/\mathbb{Z}_N$ for $N = 2, 3, 4$. Due to the relation $\mathrm{ch}_2(L) = \frac{1}{2}c_1(L)^2$ for a line bundle, the second Chern characters are in general expected to be multiples of one-half. In the cases of spinorial shifts, an additional factor of one-half appears in the definition of the line bundle upon our identification (3.9) leading to multiples of 1/8 for the second Chern characters. The shortest possible shift vectors for 3d, 3e are not the ones displayed in table 1, but are spinorial ones which are obtained by subtracting the spinorial weight $(\frac{1}{2}, \ldots, \frac{1}{2})$ of SO(32), namely $-\frac{1}{6}(1^{10}, -1, 3^5)$ and $-\frac{1}{6}(1^{14}, 3^2)$, respectively.

bundles in table 18 in order to find smooth matches of the models at the orbifold point.

To start with, we use the relation

$$\left.\begin{matrix}\beta_i \\ 2\,\alpha_i\end{matrix}\right\} = 4\,(\mathrm{ch}_2(L_i) + 1) \tag{3.11}$$

for the coefficients of $SU(N_i)$ gauge factors (or $SO(2N_i)$ with spinor representations at the orbifold point) in the anomaly polynomial to determine the second Chern characters of various line bundles, and in the second step we check if these satisfy the tadpole cancellation condition (3.8). The coefficients α_i, β_i at the orbifold point are listed in tables 13 to 15. The second Chern characters computed using the relation (3.11) are displayed in table 19. Apart from these, further line bundles embedded in U(1) gauge factors can occur. Their second Chern characters cannot be determined simply from the anomaly polynomial, but from the tadpole cancellation condition and the multiplicity of fundamental representations of the non-Abelian gauge factors in the model in question.

Table 19 can be compared to the second Chern characters of some consistent smooth models listed in table 20 with their spectra displayed in table 21. The models fall into

#	Gauge group	Matter
2-6a	$SO(28)\times U(2)$	$46\,(\mathbf{1},\mathbf{1})_2+10\,(\mathbf{28},\mathbf{2})_1$
2b	$SO(20)\times U(6)$	$14\,(\mathbf{1},\mathbf{15})_2+2(\mathbf{20},\mathbf{6})_1$
2c	$U(15)\times U(1)$	$2(\mathbf{105})_{2,0}+14(\mathbf{15})_{1,-1}+2(\mathbf{15})_{1,1}$
3b	$SO(22)\times U(4)\times U(1)$	$(\mathbf{22},\mathbf{4})_{1,0}+10(\mathbf{22},\mathbf{1})_{0,1}+10(\mathbf{1},\mathbf{6})_{2,0}+25(\mathbf{1},\mathbf{4})_{1,1}+(\mathbf{1},\mathbf{4})_{1,-1}$
3c	$SO(16)\times U(8)$	$10(\mathbf{1},\mathbf{28})_2+(\mathbf{16},\mathbf{8})_1$
3d	$U(10)\times U(5)\times U(1)$	$(\mathbf{45},\mathbf{1})_{2,0,0}+7(\mathbf{1},\mathbf{10})_{0,2,0}$ $+2(\mathbf{10},\mathbf{5}/\overline{\mathbf{5}})_{1,\pm1}+8(\mathbf{10},\mathbf{1})_{1,0,\pm1}+11(\mathbf{1},\mathbf{5})_{0,1,\pm1}$
3e	$U(14)\times U(2)$	$(\mathbf{91})_{2,0}+10(\mathbf{14},\mathbf{2})_{1,1}+(\mathbf{14},\overline{\mathbf{2}})_{1,-1}+25(\mathbf{1},\mathbf{1})_{0,2}$
4b	$SO(24)\times U(2)^2$	$4(\mathbf{24},\mathbf{2},\mathbf{1})_{1,0}+4(\mathbf{24},\mathbf{1},\mathbf{2})_{0,1}$ $+22(\mathbf{1},\mathbf{1},\mathbf{1})_{2,0}+22(\mathbf{1},\mathbf{1},\mathbf{1})_{0,2}$ $+20(\mathbf{1},\mathbf{2},\mathbf{2}/\overline{\mathbf{2}})_{1,\pm1}$
4c	$SO(20)\times U(2)\times U(4)$	$6(\mathbf{20},\mathbf{2},\mathbf{1})_{1,0}+16(\mathbf{1},\mathbf{2},\mathbf{4}/\overline{\mathbf{4}})_{1,\pm1}$ $6(\mathbf{1},\mathbf{1},\mathbf{6})_{0,2}+30(\mathbf{1},\mathbf{1},\mathbf{1})_{2,0}$
4d	$SO(16)\times U(2)\times U(6)$	$6(\mathbf{1},\mathbf{1},\mathbf{15})_{0,2}+(\mathbf{1},\mathbf{1},\mathbf{1})_{2,0}$ $12(\mathbf{1},\mathbf{2},\mathbf{6}/\overline{\mathbf{6}})_{1,\pm1}+4(\mathbf{16},\mathbf{2},\mathbf{1})_{1,0}$
4e	$U(6)\times U(10)$	$4(\mathbf{15},\mathbf{1})_{2,0}+4(\mathbf{1},\mathbf{45})_{0,2}+2(\mathbf{10},\mathbf{6}/\overline{\mathbf{6}})_{1,\pm1}$
4e'	$U(6)\times U(9)\times U(1)$	$8(\mathbf{15},\mathbf{1})_{2,0,0}+2(\mathbf{6},\mathbf{9}/\overline{\mathbf{9}},\mathbf{1})_{1,\pm1,0}$ $+10(\mathbf{6},\mathbf{1},\mathbf{1})_{1,0,\pm1}+6(\mathbf{1},\mathbf{9},\mathbf{1})_{0,1,\pm1}$
4f	$U(4)\times U(10)\times U(2)$	$8(\mathbf{6},\mathbf{1},\mathbf{1})_{2,0,0}+16(\mathbf{1},\mathbf{1},\mathbf{1})_{0,0,2}+$ $2(\mathbf{4},\mathbf{10}/\overline{\mathbf{10}},\mathbf{1})_{1,\pm1,0}+10(\mathbf{4},\mathbf{1},\mathbf{2}/\overline{\mathbf{2}})_{1,0,\pm1}+6(\mathbf{1},\mathbf{10},\mathbf{2}/\overline{\mathbf{2}})_{0,1,\pm1}$
4k	$U(15)\times U(1)$	$(\mathbf{105})_{2,0}+23\,(\mathbf{15})_{1,\pm1}$

Table 21: Massless spectra for smooth SO(32) compactifications. The supergravity, universal tensor and twenty neutral hyper multiplets are not listed. Only the overall number of non-Abelian charges is counted, i.e. $16(\mathbf{15})_{1,\pm1}$ denotes a net number of 16 hyper multiplets transforming as $\mathbf{15}$ with an arbitrary decomposition into U(1) charge assignments $(1,1)$ or $(1,-1)$, $x(\mathbf{15})_{1,1}+(16-x)(\mathbf{15})_{1,-1}$. In the same spirit $\mathbf{5}/\overline{\mathbf{5}}$ means that the multiplet transforms either in the fundamental or its conjugate representation and this difference is not specified by the second Chern characters, but the first Chern classes (which are not computable via the anomaly polynomial) of the line bundles are required.

several categories:

- One clearly sees the matching of all non-Abelian charges for models 3-6a, 3c and 4k with the orbifold case. In the cases 3-6a and 3c, just one line bundle is sufficient, whereas the matching of 4k requires two line bundles with different second Chern characters.

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- For 2a, 3e and 4b the matching works nicely when an SU(2) factor at the orbifold point is ignored.[13] In the first two cases, one line bundle is embedded. For example for 3e, $\mathrm{ch}_2(L_2) = 9\,\mathrm{ch}_2(L_1)$ leads to the natural identification $L_2 = L_1^3$, which is the expected value using the shift vector of minimal length, $\frac{1}{3}(1^{14}, 0^2) - (\frac{1}{2}, \ldots, \frac{1}{2}) = -\frac{1}{6}(1^{14}, 3^2)$.

  In case of model 4b, two independent line bundles with the same second Chern character are needed; otherwise $\mathrm{U}(2)^2$ would be enhanced to U(4).

- Models 2b, 4c, 4d, 4e, 4f match provided a breaking

$$\mathrm{SO}(2M) \to \mathrm{U}(M) \tag{3.12}$$

  occurs (for the decompositions of representations see appendix A), where 2b and 4c are realized by the embedding of one line bundle, and 4d, 4e, 4f require (at least) two different bundles. In contrast to the other cases, the SO(12) (SO(8)) gauge group in 4e (4f) stems from zero entries in the shift vector and the requirement of an U(6) (U(4)) factor instead can only be seen by the existence of the $\mathbf{32}_+$ ($\mathbf{8}_\pm$) representation in the T^2 (T and T^2) sector at the orbifold point.

- 2c and 3b match with one line bundle embedded upon the breaking

$$\mathrm{SU}(N+1) \to \mathrm{SU}(N) \times \mathrm{U}(1) \tag{3.13}$$

  suggested by the number of identical entries in the shift vectors. For example, for 2c the identifications implies $\frac{1}{4}(1^{15}, -3) \to (L, \ldots, L, L^{-3})$ (decompositions of representations are again given in appendix A), and the spectrum contains two hyper multiplets in the antisymmetric and 14+2=16 in the fundamental representation of SU(15), while at the orbifold two antisymmetric and 16 fundamentals of SU(16) appear.

- In cases 3d, 4e', both types of breakings (3.12) and (3.13) are needed to find a smooth match. The breaking of the SO(12) gauge group in 4e' is again not visible from the zero entries of the shift vectors, and at least two different line bundles have to be embedded in each model.

- For model 4a', smooth matches are possible with two line bundles embedded in $\mathrm{SO}(28) \times \mathrm{U}(1)^2$, but since $\sum_{i=1}^{2} \mathrm{ch}_2(L_i) = -24$ is the only model building constraint, we do not display any spectrum. Similarly, model 4j should have a smooth match with L_1, L_2, L_3 embedded in $\mathrm{U}(14) \times \mathrm{U}(1)^2$.

- Models 4g, 4h, 4i require some $\mathrm{SU}(6) \to \mathrm{SU}(5) \times \mathrm{U}(1)$ breaking due to the existence of **20** representations in the T^2 sector which is not evident in the shift vector with six

[13] The smooth embeddings 2-6a presented here have U(1) bundles and are thus no "standard embeddings" on $K3$. A standard embedding of the Z_2 model with SU(2) bundle will be presented in [25]. In that paper also an alternative blow-up of model 2a with $U(1)$-fluxes is presented, with flux embedded as $(L^2, L, L, 0^{13})$ with $ch_2(L) = -4$ and gauge group $\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{SO}(26)$. The latter corresponds to a different blow-up of the same orbifold model, with a different gauge symmetry breaking/enhancement.

identical entries, similarly smooth matches to 4l, 4m, 4n require some breaking of the type (3.13) due to the existence of fields transforming only in the fundamental, not bi-fundamental representation of the non-Abelian gauge factors. 6b can only be matched provided both orthogonal groups are broken along the lines of equation (3.12) and afterwards some breaking of the type (3.13) occurs.

In summary, for ten models we find smooth matches with just one line bundle embedded in SO(32) fitting with the identification of shift vectors and line bundle embeddings. In the other cases, the smooth matches are more involved. For example, the other explicitly worked out $T^4/\mathbb{Z}_4$ matches require two independent line bundles, which suggests a correspondence between the number of orbifold twist sectors and line bundles embedded.

3.3 U(1) bundles inside E_8

In section 3.2, we have shown that the shift vector of the SO(32) heterotic orbifold has a direct interpretation in terms of the embedding of line bundles. In order to transfer the argument to the $E_8 \times E_8$ case, we consider the following successive breaking

$$\begin{array}{c} E_8 \\ \mathbf{248} \end{array} \to \begin{array}{c} \mathrm{SO}(16) \\ \begin{pmatrix} \mathbf{120} \\ \mathbf{128} \end{pmatrix} \end{array} \to \begin{array}{c} \mathrm{SO}(14)\times\mathrm{U}(1) \\ \begin{pmatrix} (\mathbf{91})_0 + (\mathbf{1})_0 + [(\mathbf{14})_2 + c.c.] \\ (\mathbf{64})_1 + c.c. \end{pmatrix} \end{array}$$

$$\to \begin{array}{c} \mathrm{SO}(12)\times\mathrm{U}(1)^2 \\ \begin{pmatrix} (\mathbf{66})_{0,0} + 2(\mathbf{1})_{0,0} \\ +[(\mathbf{12})_{2,0} + (\mathbf{12})_{0,2} + (\mathbf{1})_{2,2} + (\mathbf{1})_{2,-2} + c.c.] \\ (\mathbf{32}_+)_{1,1} + (\mathbf{32}_-)_{1,-1} + c.c. \end{pmatrix} \end{array} \to \dots.$$

These breakings can be cast into the compact notation ($2 \leqslant N \leqslant 6$)

$$\begin{aligned} E_8 &\to \mathrm{SO}(2N)\times\mathrm{U}(1)^{8-N} \\ \mathbf{248} &\to (\mathbf{Adj}_{\mathrm{SO}(2N)})_0 + (8-N)\times(\mathbf{1})_0 + (\mathbf{2N})_{\underline{\pm 2, 0^{7-N}}} + (\mathbf{1})_{\underline{\pm 2, \pm 2, 0^{6-N}}} \\ &\quad + \sum_{k=0}^{[4-\frac{N}{2}]} (\mathbf{2}_+^{N-1})_{\underline{1^{8-N-2k}, -1^{2k}}} + \sum_{k=0}^{[\frac{7-N}{2}]} (\mathbf{2}_-^{N-1})_{\underline{1^{7-N-2k}, -1^{1+2k}}}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} E_8 &\to \mathrm{U}(1)^8 \\ \mathbf{248} &\to 8(\mathbf{1})_0 + (\mathbf{1})_{\underline{\pm 2, \pm 2, 0^6}} + \sum_{k=0}^{4} (\mathbf{1})_{\underline{1^{8-2k}, -1^{2k}}}, \end{aligned}$$

where underlining of the charges denotes all possible permutations. The U(1) charge assignments serve, analogously to the SO(32) case, as a guideline to the correct assignment of bundles, namely the i^{th} charge entry n_i in (3.14) corresponds to the bundle $L_i^{n_i}$.

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#	rep.
$-\chi(L_1^2)$	$(\mathbf{12})_{2,0}$
$-\chi(L_2^2)$	$(\mathbf{12})_{0,2}$
$-\chi(L_1\otimes L_2)$	$(\mathbf{32}_+)_{1,1}$
$-\chi(L_1\otimes L_2^{-1})$	$(\mathbf{32}_-)_{1,-1}$
$-\chi(L_1^2\otimes L_2^2)$	$(\mathbf{1})_{2,2}$
$-\chi(L_1^2\otimes L_2^{-2})$	$(\mathbf{1})_{2,-2}$

Table 22: Matter multiplicities for L_1, L_2 embedded in $\mathrm{SO}(12)\times \mathrm{U}(1)^2 \subset E_8$.

The tadpole contribution for all successive breakings is computed from

$$\mathrm{tr}_{E_8}\overline{F}^2 = \frac{1}{30}\mathrm{Tr}_{E_8}\overline{F}^2 = 16\pi^2\cdot 4\sum_{i=1}^{N}\mathrm{ch}_2(L_i)$$

with all charge assignments integer valued and the corresponding integer powers of line bundles associated as for the following example.

Consider for concreteness the embedding of two line bundles L_1, L_2 inside an E_8 factor,

$$(L_1, L_2, 0^6). \tag{3.15}$$

The resulting massless spectrum consists of the gauge group $\mathrm{SO}(12)\times\mathrm{U}(1)^2_{\mathrm{massive}}$ and matter with the multiplicities listed in table 22.

Similar to the SO(32) embeddings of line bundles, non-Abelian gauge enhancements occur for special combinations of several line bundles. Consider for concreteness the case which corresponds to the so called orbifold "standard embedding". The spectrum in table 22 contains states $(\mathbf{32}_\mp)_{1,\mp1}$ and $(\mathbf{1})_{2,\mp2}$ transforming as the trivial line bundle $\mathcal{O}$ for $L_2 = L_1^{\pm1}$ thereby leading to the gauge enhancement

$$\mathrm{SO}(12)\times\mathrm{U}(1)_{\mathrm{massless}}\times\mathrm{U}(1)_{\mathrm{massive}} \to \mathrm{SO}(12)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\mathrm{massive}} \to E_7\times\mathrm{U}(1)_{\mathrm{massive}}, \tag{3.16}$$

where in the first step, one observes that $\mathrm{U}(1)_{\mathrm{massless}} = \mathrm{U}(1)_1 \mp \mathrm{U}(1)_2$ remains massless due to the linear dependence of the two line bundles, and the states in the $(\mathbf{1})_{2,\mp2}$ representation lead to $\mathrm{U}(1)_{\mathrm{massless}} \to \mathrm{SU}(2)$. In the second step, the vectors transforming as $(\mathbf{32}_\mp, \mathbf{2})_0$ provide the E_7 enhancement.

The two choices $L_2 = L_1^{\pm1}$ differ only in their fulfillment of the K-theory constraint,

$$c_1(W_{\mathrm{total}}) = c_1(L_1) + c_1(L_2) = \begin{cases} 2\,c_1(L_1) & L_2 = L_1 \\ 0 & L_2 = L_1^{-1} \end{cases}$$

in analogy to the SO(32) case (3.10).

The resulting multiplicities of the spectrum are listed in table 23.

The same line of argument applies to other cases, those who are relevant for the matching of $T^4/\mathbb{Z}_N$ orbifold spectra are listed in tables 23, 24 and 25. Notice that

		SO(14) × U(1)	E_7 × U(1)	E_6 × SU(2) × U(1)	SU(8) × U(1)
#V	#H	$(L,0^7)$ $(L^{1/4},\dots,L^{1/4},L^{-3/4})$	$(L^{1/2},L^{1/2},0^6)$ $(L^{1/4},\dots,L^{1/4})$	$(L^{1/2},L^{1/2},L,0^5)$ $(L^{1/2},\dots,L^{1/2},0^2)$	$(L^{1/2},L^{1/2},L^{1/2},L^{1/2},L,0^3)$ $(L^{1/2},\dots,L^{1/2},L^{-1/2})$ $(L^{1/4},\dots,L^{1/4},L^{5/4})$
1	0	$(\mathbf{91})_0$	$(\mathbf{133})_0$	$(\mathbf{78},\mathbf{1})_0$	$(\mathbf{63})_0$
1	0	-	-	$(\mathbf{1},\mathbf{3})_0$	-
1	0	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$
0	$-\chi(L)$	$(\mathbf{64})_1$	$(\mathbf{56})_1$	$(\overline{\mathbf{27}},\mathbf{2})_1$	$(\mathbf{56})_1$
0	$-\chi(L^2)$	$(\mathbf{14})_2$	$(\mathbf{1})_2$	$(\mathbf{27},\mathbf{1})_2$	$(\mathbf{28})_2$
0	$-\chi(L^3)$	-	-	$(\mathbf{1},\mathbf{2})_3$	$(\mathbf{8})_3$

Table 23: General spectrum from embedding a line bundle L inside one E_8 factor. The embedding is specified by the shift vector corresponding to the Cartan generators. In all cases, where the shift vector contains at least one zero entry, the signs of the exponents of the line bundle entries are arbitrary, and for shortness only positive exponents are listed. The powers of the line bundles are rescaled such that the smallest multiplicity of a matter state is computed from $-\chi(L)$. Part 1.

		SO(10)×SU(3)×U(1)	SU(7)×SU(2)×U(1)	SU(5)×SU(3)×SU(2)×U(1)	SO(12)×U(1)^2
#V	#H	$(L,L,L,0^5)$	$(L^{1/2},\dots,L^{1/2},L^{3/2},0^2)$ $(L^{1/4},\dots,L^{1/4},L^{-7/4})$	$(L^{1/2},\dots,L^{1/2},L^{5/2},0^2)$	$(L^{1/2},L^{3/2},0^6)$
1	0	$(\mathbf{45},\mathbf{1})_0$	$(\mathbf{48},\mathbf{1})_0$	$(\mathbf{24},\mathbf{1},\mathbf{1})_0$	$(\mathbf{66})_{0,0}$
1	0	$(\mathbf{1},\mathbf{8})_0$	$(\mathbf{1},\mathbf{3})_0$	$(\mathbf{1},\mathbf{8},\mathbf{1})_0$ $(\mathbf{1},\mathbf{1},\mathbf{3})_0$	$(\mathbf{1})_{0,0}$
1	0	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_0$	$(\mathbf{1})_{0,0}$
0	$-\chi(L)$	$(\overline{\mathbf{16}},\overline{\mathbf{3}})_1$	$(\mathbf{21},\mathbf{2})_1$	$(\mathbf{5},\mathbf{3},\mathbf{2})_1$	$(\mathbf{12})_{3,1}$ $(\mathbf{32}_-)_{-2,1}$
0	$-\chi(L^2)$	$(\mathbf{10},\mathbf{3})_2$	$(\overline{\mathbf{35}},\mathbf{1})_2$	$(\mathbf{10},\overline{\mathbf{3}},\mathbf{1})_2$	$(\mathbf{1})_{-4,2}$ $(\mathbf{32}_+)_{1,2}$
0	$-\chi(L^3)$	$(\mathbf{16},\mathbf{1})_3$	$(\overline{\mathbf{7}},\mathbf{2})_3$	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{2})_3$	$(\mathbf{12})_{-1,3}$
0	$-\chi(L^4)$	$(\mathbf{1},\overline{\mathbf{3}})_4$	$(\mathbf{7},\mathbf{1})_4$	$(\overline{\mathbf{5}},\mathbf{3},\mathbf{1})_4$	$(\mathbf{1})_{2,4}$
0	$-\chi(L^5)$	-	-	$(\mathbf{1},\overline{\mathbf{3}},\mathbf{2})_5$	-
0	$-\chi(L^6)$	-	-	$(\mathbf{5},\mathbf{1},\mathbf{1})_6$	-

Table 24: Embedding a line bundle L inside E_8. Part 2. In the last column, the charge assignments are $(Q_{\text{massless}},Q_{\text{massive}}) = (\frac{3Q_1-Q_2}{2},\frac{Q_1+3Q_2}{2})$ in terms of the original charges.

these tables are valid also for Calabi-Yau compactification, since the dimension of the space affects only the expansion of $\chi(L)$, which for four dimensional models is $\chi(L)_{CY_3} = \int_{CY_3}\big(\mathrm{ch}_3(L)+\frac{1}{12}c_2(CY_3)\,c_1(L)\big)$.

With the help of the counting of multiplicities in tables 23 to 25, the general form of the non-Abelian part of the anomaly polynomial can be computed to be of the shape (2.26) with the coefficients a_r listed in table 26 and $\tilde{a}_r$ as in the orbifold case (2.27). As for the SO(32) matchings, the coefficients a_r of the non-Abelian gauge factors will serve in section 3.3.1

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#V	#H	$SU(5)\times SU(4)\times U(1)$	#H	$(L,L,L,L,L,0^3)$ $(L^{1/2},L^{1/2},L^{1/2},L^{1/2},L^2,0^3)$
1	0	$(\mathbf{24},\mathbf{1})_0$	$-\chi(L^3)$	$(\mathbf{5},\overline{\mathbf{4}})_3$
1	0	$(\mathbf{1},\mathbf{15})_0$	$-\chi(L^4)$	$(\mathbf{10},\mathbf{1})_4$
0	$-\chi(L)$	$(\mathbf{10},\mathbf{4})_1$	$-\chi(L^5)$	$(\mathbf{1},\mathbf{4})_5$
0	$-\chi(L^2)$	$(\mathbf{5},\mathbf{6})_2$		

Table 25: Embedding a line bundle L inside E_8. Part 3.

G_r	a_r
SO(12)	$-(5\mathrm{ch}_2(L)+6)$
SO(14)	$-2(\mathrm{ch}_2(L)+3)$
E_7	$-(\frac{1}{6}\mathrm{ch}_2(L)+1)$
$(E_6,\mathrm{SU}(2))$	$-(\mathrm{ch}_2(L)+2)\times(1,6)$
SU(8)	$-4(2\mathrm{ch}_2(L)+3)$
(SO(10), SU(3))	$-6\,(\mathrm{ch}_2(L)+1)\times(1,2)$
(SU(7), SU(2))	$-(14\mathrm{ch}_2(L)+12)\times(1,1)$
(SU(5), SU(3), SU(2))	$-6\,(5\mathrm{ch}_2(L)+2)\times(1,1,1)$
(SU(5), SU(4))	$-(20\mathrm{ch}_2(L)+12)\times(1,1)$

Table 26: Relation of coefficients in the $E_8\times E_8$ anomaly polynomials and second Chern characters.

as the guideline to compute the second Chern characters of the smooth models from the orbifold data, whereas the role of the U(1) factors at the orbifold point cannot be recovered by the smooth ansatz.

3.3.1 Explicit $K3$ realizations of $E_8\times E_8$ orbifold spectra

In this section, we give an explicit matching of some $E_8\times E_8$ orbifold spectra by smooth $K3$ compactifications with one line bundle and comment on obstructions for other cases.

As for the SO(32) case, the natural identification of $\mathbb{Z}_N$ orbifold shift vectors and line bundles is

$$\frac{1}{N}(1,\dots,1,n,0,\dots,0)\to(L,\dots,L,L^n,0,\dots,0),$$

and in order to compute the second Chern characters of the line bundles, the correspondence between orbifold coefficients a_r of non-Abelian gauge factors in the anomaly polynomial and second Chern characters in table 26 is used. The result is obtained from the larges non-Abelian gauge factor and listed in the fourth column of table 16 in section 2.4, where the powers of the line bundle L from which the Chern characters are computed are those given in tables 23 to 25. The resulting values for $\mathrm{ch}_2(L)$ take the form of fractional numbers for model IIIe and several of those based on $T^4/\mathbb{Z}_N$ orbifolds for $N=4,6$. Those are the

#	Embedding	$\mathrm{ch}_2(L)$	$\mathrm{ch}_2(\tilde{L})$
II-VIa	$(L^{1/2}, L^{1/2}, 0^6; 0^8)$	-12	0
IIb	$(L, 0^7; \tilde{L}^{1/2}, \tilde{L}^{1/2}, 0^6)$	-4	-4
IIIb	$(L, 0^7; \tilde{L}, 0^7)$	-3	-3
IIIc	$(L^{1/2}, L^{1/2}, L^{1/2}, L^{1/2}, L, 0^3; 0^8)$	-3	0
IIId	$(L^{1/2}, L^{1/2}, L, 0^5; \tilde{L}^{1/2}, \tilde{L}^{1/2}, 0^6)$	-3	-3
IVb	$(L^{1/2}, L^{1/2}, 0^6; \tilde{L}^{1/2}, \tilde{L}^{1/2}, 0^6)$	-6	-6
IVc	$(L^{1/2}, L^{1/2}, 0^6; \tilde{L}, 0^7)$	-8	-2
IVe	$(L^{1/2}, L^{1/2}, L, 0^5; \tilde{L}, 0^7)$	-2	-3

Table 27: Line bundle embeddings for some smooth matches of orbifold spectra.

ones which have hyper multiplets in the twisted orbifold sectors with charges under the remainders of *both* E_8 gauge factors and cannot be reproduced by our smooth ansatz with just one line bundle inside each E_8. The embeddings which have smooth matches along our general simple rules are given in table 27 with their spectra in table 28.

As for the SO(32) cases, there is a perfect match of all non-Abelian charges for models III-VIa and IIIb, and for IIa and IVb the massless spectra agree up to an additional SU(2) gauge factor at the orbifold point. Models IIIc and IIId match at the non-Abelian level when decomposing $\mathrm{SU}(N+1) \to \mathrm{SU}(N) \times \mathrm{U}(1)$, and IVc matches upon the breaking $\mathrm{SO}(16) \to \mathrm{SO}(14) \times \mathrm{U}(1)$.[14] Model IIb requires the same breaking, and additionally the SU(2) gauge factor is only present at the orbifold point. Finally, IVe has a smooth match, but the SU(2) representations don't agree. This mismatch is due to the mixing with a non-perturbative SU(2) symmetry of the orbifold background [23].

Except for IVe, all second Chern characters in table 27 are consistent with embedding (multiples of) the same line bundle in both E_8 gauge factors. Correspondingly, one linear combination of the Abelian gauge groups will stay massless. The U(1) charges given in table 28 are, however, the original ones. As an example, in model IVb, the massive and massless charges are proportional to $(Q_1 \pm Q_2)$.

Similarly to the SO(32) matches, one line bundle is not sufficient for many $T^4/\mathbb{Z}_N$ models with $N = 4, 6$. As another example besides IVe, consider the shift vector $\frac{1}{4}(1, 3, 0^6)$. The ansatz $(L^{1/2}, L^{3/2}, 0^6)$ does not provide the correct number of $\mathbf{32}_+$ and $\mathbf{32}_-$ spinor representations of SO(12). Instead, by comparison with the multiplicities of the fundamental and spinor representations in the orbifold spectra we obtain the following constraints

[14]The matching for this breaking is best seen by comparing the (for this model vanishing) number of spinor representations: $\mathbf{128} \to \mathbf{64} + \overline{\mathbf{64}}$. As displayed in appendix A, the adjoint representation of higher rank contains two fundamental representation which together with two hyper multiplets group into massive vectors upon symmetry breaking. Therefore 8-2=6 is the correct number of $\mathbf{14}$'s in the smooth model.

#	Gauge Group	Spectrum
II-VIa	$E_7\times\mathrm{U}(1)\times E_8$	$10(\mathbf{56})_2+46(\mathbf{1})_4$
IIb	$\mathrm{SO}(14)\times\mathrm{U}(1)\times E_7\times\mathrm{U}(1)$	$14(\mathbf{14},\mathbf{1})_{2,0}+2(\mathbf{64},\mathbf{1})_{1,0}$ $+2(\mathbf{1},\mathbf{56})_{0,1}+14(\mathbf{1})_{0,2}$
IIIb	$\mathrm{SO}(14)\times\mathrm{U}(1)\times\mathrm{SO}(14)\times\mathrm{U}(1)$	$(\mathbf{64},\mathbf{1})_{1,0}+(\mathbf{1},\mathbf{64})_{0,1}$ $+10(\mathbf{14},\mathbf{1})_{2,0}+10(\mathbf{1},\mathbf{14})_{0,2}$
IIIc	$\mathrm{SU}(8)\times\mathrm{U}(1)\times E_8$	$(\mathbf{56})_1+10(\mathbf{28})_2+25(\mathbf{8})_3$
IIId	$E_6\times\mathrm{SU}(2)\times\mathrm{U}(1)\times E_7\times\mathrm{U}(1)$	$(\mathbf{27},\mathbf{2};\mathbf{1})_{1,0}+10(\mathbf{27},\mathbf{1};\mathbf{1})_{2,0}+25(\mathbf{1},\mathbf{2};\mathbf{1})_{3,0}$ $+(\mathbf{1},\mathbf{1};\mathbf{56})_{0,1}+10(\mathbf{1})_{0,2}$
IVb	$E_7\times\mathrm{U}(1)\times E_7\times\mathrm{U}(1)$	$4(\mathbf{56};\mathbf{1})_{1,0}+4(\mathbf{1};\mathbf{56})_{0,1}+22(\mathbf{1})_{2,0}+22(\mathbf{1})_{0,2}$
IVc	$E_7\times\mathrm{U}(1)\times\mathrm{SO}(14)\times\mathrm{U}(1)$	$6(\mathbf{56};\mathbf{1})_{1,0}+30(\mathbf{1})_{2,0}+6(\mathbf{1};\mathbf{14})_{0,1}$
IVe	$E_6\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SO}(14)\times\mathrm{U}(1)$	$6(\mathbf{27},\mathbf{1};\mathbf{1})_{2,0}+16(\mathbf{1},\mathbf{2};\mathbf{1})_{3,0}$ $+(\mathbf{1},\mathbf{1};\mathbf{64})_{0,1}+10(\mathbf{1},\mathbf{1};\mathbf{14})_{0,2}$

Table 28: Some perturbative smooth $E_8\times E_8$ spectra.

on two different line bundles embedded as in table 22:

	IVg	IVh	IVi	IVl	VIb
$\mathrm{ch}_2(L_1)+\mathrm{ch}_2(L_2)$	-6	-5	-4	$-\frac{5}{2}$	-3
$c_1(L_1)\,c_1(L_2)$	0	-3	-2	$-\frac{1}{2}$	0

Model IVg has then gauge group $\mathrm{SO}(12)\times\mathrm{U}(1)^2_{\text{massive}}\times E_8$ with $4(\mathbf{32}_+)_{1,1}+4(\mathbf{32}_-)_{1,-1}+x(\mathbf{12})_{2,0}+(20-x)(\mathbf{12})_{0,2}+22(\mathbf{1})_{2,2}+22(\mathbf{1})_{2,-2}$, where x depends on the value of $\mathrm{ch}_2(L_1)$. Models IVh, IVi and VIb nicely work along the same lines. For the remaining cases, the second Chern characters in table 16 are fractional and the matching is more complicated involving also splittings into several line bundles for other shift vectors.

In general, the gauge symmetry breaking is more involved than in model IVg where one has just an additional massive $\mathrm{U}(1)$ factor. For example, the assignment of two line bundles $(L_1,L_1,L_2,0^5)$ leads to the gauge group $\mathrm{SO}(10)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2_{\text{massive}}$. For $L_2=L_1$, the second factor is enhanced to $\mathrm{SU}(3)$ as displayed in table 24, whereas for $L_2=L_1^2$ the first factor is enhanced to E_6 as shown in table 23.

Since $T^4/\mathbb{Z}_N$ orbifolds with $N=4,6$ have more than one twist sector and more than one kind of fixed point, it is not surprising that more than one line bundle is required to obtain smooth matches.

4. Towards explicit realizations of line bundles

Throughout this article, we are working with line bundles whose second Chern characters are determined via the matching of the anomaly polynomials and the tadpole cancellation constraint. In this section, we speculate on the explicit realization of these line bundles.

The naive starting point is motivated by S-duality with type II compactifications, namely in the orbifold limit $T^4/\mathbb{Z}_2$, one starts with

$$c_1(L) = \frac{1}{2}\sum_{i=1}^{16} E_i \quad \Rightarrow \quad \mathrm{ch}_2(L) = -4, \tag{4.1}$$

where i labels the orbifold fixed points and E_i with $E_i \cdot E_j = -2\delta_{ij}$ the two forms associated to the blown down two-cycles at the orbifold point. The background gauge field is localized at the orbifold fixed points and democratically distributed among them. The bundle ansatz (4.1) gives the correct second Chern character for models 2b and IIb. Since 2c is based on a spinorial shift, one can speculate that the factor one-half directly enters the definition of the line bundle,

$$c_1(L_{2\mathrm{c}}) = \frac{1}{4}\sum_{i=1}^{16} E_i,$$

which indeed gives the desired second Chern character.

In the same spirit, one can make the ansatz for a democratic distribution over the nine $T^4/\mathbb{Z}_3$ fixed points,

$$c_1(L) = \frac{1}{3}\sum_{i=1}^{9}\left(E_i^{(1)} - E_i^{(2)}\right) \quad \Rightarrow \quad \mathrm{ch}_2(L) = -3, \tag{4.2}$$

with the intersection form equal to minus the Cartan matrix of A_2, i.e. $E_i^{(1)} \cdot E_j^{(1)} = E_i^{(2)} \cdot E_j^{(2)} = -2\delta_{ij}$, $E_i^{(1)} \cdot E_j^{(2)} = \delta_{ij}$. This fits with the second Chern characters of models 3b, 3c, IIIb - IIId, and again one finds the matching with the spinorial shift 3e upon multiplication by one-half. The value for the standard embeddings 3a and IIIa is obtained by multiplying by two.

The cases $T^4/\mathbb{Z}_N$ for $N = 4, 6$ are more involved due to the occurrence of different types of exceptional cycles at various singularities as discussed in section 2.

Even though this ansatz fits nicely for several models, it is not at all obvious that this is indeed the correct correspondence since the partition functions mix contributions from the space-time and gauge sector embeddings. This it what makes the transfer of our $T^4/\mathbb{Z}_N$ ansatz to $T^6/\mathbb{Z}_N$ compactifications so difficult: whereas $K3$ is unique, out of the multitude of Calabi-Yau threefolds it is not clear if the blown-up model will have the same Hodge numbers as the $T^6/\mathbb{Z}_N$ orbifold background.

5. Flat directions and blow-up of the orbifold models

In the previous sections, the matching between orbifold models and smooth K3 compactifications has been shown. The matching is satisfactory only up to some caveats, summarized in section 3.2.1 for the SO(32) case (the E_8 case can be treated in a similar fashion as discussed in 3.3.1). The main point is a reduction of the gauge symmetry in the matching, of the type $\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1)$ or $\mathrm{SO}(2N) \to \mathrm{SU}(N) \times \mathrm{U}(1)$, accompanied by the reduction of the total rank of the gauge group due to the "disappearance" of a number of U(1) factors equal to the number of bundles introduced on the smooth model building side of the matching. From the latter perspective these U(1) factors are anomalous and get a mass term due to a modified Green-Schwarz mechanism.

Such a gauge symmetry reduction is completely natural. Indeed, the matching between orbifold models and smooth models is meaningful only if we consider a blow-up of the orbifold singularities. This corresponds to the fact that some of the twisted states acquire a non trivial vacuum expectation value (vev). Since, generically, twisted states are charged under the gauge symmetry, the blow up corresponds to a modified Higgs mechanism.

The requirement that supersymmetry is preserved in the blow up implies that the "allowed" blow up directions are flat directions of the moduli potential. In the following we investigate the properties of such flat directions and show the details of the blow up in the SO(32) and $E_8 \times E_8$ models.

We avoid the complete study of the supergravity description of the models, that would clarify the precise matching of the models beyond the spectrum point of view, and would spread new light also on the fate of the massive gauge symmetries, but that goes well beyond the purposes of present paper.

We also comment that, typically, many flat directions are present in an orbifold model, and we expect this to be true also for the corresponding smooth model. In this sense, given each smooth model, we search in the moduli field configuration of its orbifold companion, in order to find a vacuum (among the others) where the matching is complete. A clear improvement of this picture would be a complete map between the moduli spaces of each model, but such a result is unfortunately not available at the moment.

5.1 D-flatness and blow up of K3 orbifold models

In a $d = 6$ $\mathcal{N} = 1$ model, the potential for the hyper multiplets is completely determined by the gauge interactions. It contains only terms arising from the integration of auxiliary fields in the vector superfield, and in this sense it is called "D-term potential", even though from a $d = 4$ perspective it contains both the D-term of the $d = 4$ vector and the F-term of the chiral field that combines with the $d = 4$ vector to form the $d = 6$ vector. We assume that the kinetic function of the hyper multiplets is canonically normalized at the orbifold point, we comment later about perturbations of this assumption.

Given an hyper multiplet with label i we can organize the four real fields in a complex doublet Φ_i, then, given σ^a the three Pauli matrices, and t^α the generators of the gauge group we define

$$D^{a,\alpha} = \Phi_i^\dagger \sigma^a t_{ij}^\alpha \Phi_j \tag{5.1}$$

and the scalar potential in six dimensions is then

$$V = \sum_{a,\alpha} D^{a,\alpha} D^{a,\alpha}. \tag{5.2}$$

In the presence of U(1) sectors we have $t_{ij}^\alpha = \delta_{ij} q_i$, with q_i the U(1) charge, and we just drop the α index. From a $d = 4$ perspective we can see the standard D-term, arising from

$D^{3,\alpha}$ due to $\sigma^3 = \mathrm{Diag}(1,-1)$ and the fact that a hyper multiplet contains two complex scalars with opposite charges. Moreover, there are extra terms involving $D^{1,\alpha}$ and $D^{2,\alpha}$. As stated above, the latter are, from a $d = 4$ perspective, the F-term of the chiral multiplet that mixes with the $d = 4$ vector multiplet to form the $d = 6$ vector multiplet.

The existence of flat directions depends on the field content of the model. We check now the minimal content necessary to build flat directions, and the kind of gauge symmetry breaking that is produced when a vev is switched on along these directions. The study of the general case goes beyond our scope, since the models we studied contain only very specific representations of $\mathrm{U}(N)$, $\mathrm{SO}(2N)$ and exceptional gauge groups, and only some of them are necessary to explain the blow-ups, namely the fundamental representations of $\mathrm{U}(N)$ and $\mathrm{SO}(2N)$ groups and the spinorial representation of $\mathrm{SO}(2N)$. After the description of each flat direction we show how it is possible to switch it on in some specific example and accommodate the matching of the related orbifold model with the smooth case. We notice that the matching can be made perfect in each of the models, nevertheless, we also notice that not always the "minimal" blow up possibility is enough to achieve such a result.

5.2 D-flatness in the $\mathrm{U}(1)$ case: accommodated matching in the 3-6a, 3c and III-VIa, IIIb models

In the presence of a $\mathrm{U}(1)$ symmetry and charged non-Abelian singlets in the spectrum, a flat direction can exist only in case there are at least two such singlets, independently on the charge. Indeed, in case only one singlet is present, with charge c and scalars $\Phi = (\phi_x, \phi_y)$, contrarily to what one would expect, there is no flat direction. The naive expectation arises from the fact that the two scalars have opposite $\mathrm{U}(1)$ charges and $D^3 = c(|\phi_x|^2 - |\phi_y|^2)$. Unfortunately, the other contributions are such that

$$V = c^2(|\phi_x|^2 + |\phi_y|^2)^2, \tag{5.3}$$

and there are no flat directions.

In the presence of two hyper multiplets $\Phi_1 = (\phi_{1x}, \phi_{1y})$, $\Phi_2 = (\phi_{2x}, \phi_{2y})$ of charge q_1 and q_2 we have instead

$$V = \left[q_1(|\phi_{1x}|^2 + |\phi_{1y}|^2) + q_2(|\phi_{2x}|^2 + |\phi_{2y}|^2)\right]^2 - 4q_1q_2|\phi_{1x}\phi_{2y} - \phi_{2x}\phi_{1y}|^2. \tag{5.4}$$

Given this, we can choose $q_1 = 1$, $q_2 = -1$, and the potential is a sum of positive terms

$$V = \left(|\phi_{1x}|^2 + |\phi_{1y}|^2 - |\phi_{2x}|^2 - |\phi_{2y}|^2\right)^2 + 4|\phi_{1x}\phi_{2y} - \phi_{2x}\phi_{1y}|^2. \tag{5.5}$$

The flat directions are then given by the conditions

$$|\phi_{1x}|^2 + |\phi_{1y}|^2 = |\phi_{2x}|^2 + |\phi_{2y}|^2, \;\; \phi_{1x}\phi_{2y} = \phi_{2x}\phi_{1y}. \tag{5.6}$$

The second condition completely fixes one complex field in terms of the others. Assume for a moment that ϕ_{2y} is nonzero, then

$$\phi_{1x} = \frac{\phi_{2x}\phi_{1y}}{\phi_{2y}}, \tag{5.7}$$

and replacing this in the first condition we obtain

$$\frac{|\phi_{2x}|^2 + |\phi_{2y}|^2}{|\phi_{2y}|^2}|\phi_{1y}|^2 = |\phi_{2x}|^2 + |\phi_{2y}|^2. \tag{5.8}$$

This implies that ϕ_{2x} and ϕ_{2y} can be taken in the whole space $\mathbb{C}^2$. If they are chosen away from the origin of $\mathbb{C}^2$, there is an extra condition $|\phi_{1y}| = |\phi_{2y}|$, so that ϕ_{1y} is defined up to its phase, and ϕ_{1x} is well defined in equation (5.7). If instead $\phi_{2x} = \phi_{2y} = 0$, both ϕ_{1y} and ϕ_{1x} must also be zero, given the original condition. If only ϕ_{2y} is zero and $\phi_{2x} \neq 0$, still ϕ_{1y} must be zero given the original conditions. The flat directions are then locally given by the complex plane $\mathbb{C}_1$ times the space $\mathbb{C}_2 \times S^1$, where the "radius" of S^1 depends on the value of the $\mathbb{C}_2$ "coordinate".

If a vev is switched on along one of the flat directions, the $\mathrm{U}(1)$ vector boson becomes massive. The gauge symmetry breaking is given by the kinetic terms for the fields ϕ_{ix}, ϕ_{iy}. We can rearrange these fields into a vector with four entries, $\tilde{\phi}_I$. The kinetic term is given by

$$M_{I,J}(\tilde{\phi}_K)(\mathcal{D}\tilde{\phi}_I)^\dagger \mathcal{D}\tilde{\phi}_J. \tag{5.9}$$

As mentioned above, we assume that the matrix M is positive definite at the orbifold point. We can argue that it will remain such in case a "small" vev is switched on along the flat direction.[15]

A positive mass term is generated for the $\mathrm{U}(1)$ gauge vector boson A via the term

$$M_{I,J}(\langle\tilde{\phi}_K\rangle)\langle q_I \tilde{\phi}_I^*\rangle\langle q_J \tilde{\phi}_J\rangle A^2. \tag{5.10}$$

The computation above shows that if non-Abelian singlets with $\mathrm{U}(1)$ charge are present among the twisted states, a blow up with the $\mathrm{U}(1)$ gauge vector becoming massive is actually possible, and a single combination of the singlets is "eaten" in the process, so that the matching between the spectra is achieved (we remind that the computation on the smooth side actually provides only the difference between the number of hypers and the number of massless vector multiplets, and thus if in some process a vector boson becomes massive, a corresponding hyper multiplet must disappear from the spectrum). This does not imply that no blow up is possible in the presence of a single hyper multiplet per fixed point: two hyper multiplets coming from different fixed points do produce flat directions, meaning that the independent blow up of a single fixed point is impossible, but a mutual blow up of many points is allowed.

Given this we can argue that a prefect matching between orbifold and smooth realizations in the 3-6a, 3c and III-VIa, IIIb cases can be achieved, simply by switching

[15]Strictly speaking, the ϕ dependence of M would also affect the form of the D-term potential and the D-flatness issue. Indeed, the extra corrections would in general produce new terms in the potential, of higher order in the fields ϕ. We assume that we are allowed to neglect these extra terms, since, under the assumption that $\langle\phi_i\rangle \ll 1$, they are parametrically smaller than those taken into account here. Indeed, we expect that these extra terms can at most modify the exact shape of the flat direction, without removing it.

on vevs along the flat directions given by the twisted non-Abelian singlets with U(1) charges present in the spectrum. In all the other models a similar U(1) breaking is also present, but accompanied by a rank preserving gauge symmetry breaking of the form $\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1)$ or $\mathrm{SO}(2N) \to \mathrm{SU}(N) \times \mathrm{U}(1)$, as we see in the following.

5.3 D-flatness in the SU(N) case: accommodated matching in the 2a, 2c, 3b, 3d, 3e, 4a', 4b, 4e', 4g-i and IIa, IIIc, IIId, IVb models

In the presence of an SU(N) group, there is a D-term potential corresponding to each generator of the gauge group. The condition of D-flatness is more complicated, but, for our purposes, we are allowed to consider only the case of fields in the fundamental representation $\mathbf{N}$, i.e. arrays Φ with N entries Φ_i. We consider only the case that only $\langle\Phi_1\rangle \neq 0$. In this way, only one D-flatness condition must be taken into account, completely equivalent to the one studied in the U(1) case. This implies immediately that only in the presence of at least two fields in the $\mathbf{N}$ representation, possibly also coming from different fixed points, a flat direction can be built. We conclude that the SU(N) group is broken to SU($N-1$) (nothing in the SU(2) case), and the broken vector bosons become massive. The fields in the $\mathbf{N}$ representations are decomposed into $(\mathbf{N-1}) \oplus (\mathbf{1})$, and two of the $(\mathbf{N-1})$'s plus a singlet are "eaten" in the process, consistently with the matching of the spectrum.

This mechanism can be implemented, in the 2a, 2c, 3b, 3d, 3e, 4a', 4b, 4e', 4g-i and IIa, IIIc, IIId, IVb models, to match the gauge groups and the massless spectrum.

5.4 D-flatness in the SO($2N$) case: accommodated matching in the 2b, 3d, 4c-f, 4i, 6b models

In all the 2b, 3d, 4c-f, 4i, 6b models the exact matching between the orbifold and the smooth realization requires a mechanism that breaks, on the orbifold side, some SO($2N$) group to its subgroup $\mathrm{SU}(N) \times \mathrm{U}(1)$. Such a breaking can be explained in all the models by the same mechanism, namely an Higgs mechanism for a twisted field in the spinorial representation of the SO($2N$) gauge group. We show in the following that such a mechanism can be introduced along flat directions of the potential.

It is possible to check that, among the twisted states of the models mentioned above, there are always spinorial representations that decompose under the breaking $\mathrm{SO}(2N) \to \mathrm{SU}(N) \times \mathrm{U}(1)$ as described in appendix A.

It is crucial to notice that these spinors can have negative chirality *only* in case N is odd. Thus, under the decomposition, a singlet is always present in the spectrum. A vev of such a singlet is of course responsible for the symmetry breaking, but we have to prove that such a vev can be switched on along a flat direction of the potential. In other words, we have to check the D-flatness condition for each of the gauge group generators. On the other hand, provided that we switch on only singlets, even though they arise from different spinors, we have a single condition, since the trace of any generator over the vacuum will be the same with a different weight (that is zero if the singlet is not mapped into itself by the generator). Thus, we have only copies of the same flatness condition (5.4), studied in the U(1) case in presence of at least two singlets, and so D-flatness is ensured in case the decomposition of the multiplets present in a model provides (at least) two singlets.

6. Conclusions

In this article, we have fully determined all SO(32) and $E_8 \times E_8$ heterotic orbifold spectra on $T^4/\mathbb{Z}_N$ for $N = 2, 3, 4$, i.e. for $N = 2, 3$ added the missing U(1) charges to the models of [17] and computed the $N = 4$ spectra; for $N = 6$, we have given some examples. On the smooth side of U(1) embeddings in $K3$, we have displayed a systematic treatment of E_8 line bundle embeddings and specialized on U(1) bundles in the SO(32) cases of [16].

Using the field theoretical anomaly eight-forms, we have been able to map non-Abelian gauge groups at the orbifold point to those of the smooth phase with just one line bundle. Up to the fact that at $\mathbb{Z}_2$ singularities, an additional SU(2) gauge factor can occur or the rank of some gauge factor is enhanced by one according to $\mathrm{SU}(N) \to \mathrm{SU}(N+1)$, $\mathrm{SU}(N) \to \mathrm{SO}(2N)$ or $\mathrm{SO}(2N) \to \mathrm{SO}(2N+2)$ in the orbifold limit, we find agreement for all $\mathbb{Z}_2$ spectra, all but one $\mathbb{Z}_3$ models for both SO(32) and $E_8 \times E_8$ breakings, and part of the $\mathbb{Z}_4$ models with just one line bundle embedded. In section 5, we have shown that the seeming mismatches in the orbifold point and $K3$ non-Abelian gauge groups disappear if the singularities are blown up, and the massless spectra are identical. We have argued that for the remaining $\mathbb{Z}_4$ and $\mathbb{Z}_6$ models, similar embeddings with more than one line bundle will appear, and in the blow-up procedure more than rank 1 of the gauge group will be broken. Such a conclusion is supported by the fact that only in the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases the orbifold fixed points are all equivalent, and a single line bundle, switched on "democratically" among them, can be enough for the matching: in the $\mathbb{Z}_4$ ($\mathbb{Z}_6$) case the orbifold contains two (three) different fixed point species, namely, there are both $\mathbb{Z}_4$ and $\mathbb{Z}_2$ ($\mathbb{Z}_6$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$) fixed points.

The role of the U(1) charges in all models clearly differs between the orbifold point and the smooth geometry, and the absence of any 2×6 factorization of the anomaly polynomial at the orbifold point suggests that the Abelian gauge bosons there remain all massless whereas in our class of smooth embeddings, invariably some U(1) gauge factor acquires a mass. This phenomenon is most easily seen in case of the standard embeddings 3-6a in SO(32), which at the orbifold point have the same net number of non-Abelian representations but differ in the U(1) charge assignments of the twisted sectors. All these models have the same smooth match according to our identification rule (1.1), and the field theoretical analysis of the blowing-up procedure reveals that the U(1)s acquire a mass as needed. The same applies to the $E_8 \times E_8$ standard embeddings III-VIa.

At our level of matching the non-Abelian part of the spectra, the knowledge of the second Chern characters of the line bundles is sufficient, and we have digressed only briefly on potential explicit bundle realizations.

The six-dimensional analysis presented here is a particularly well tractable set-up due to the uniqueness of $K3$ and the strong conditions on the spectrum from gravitational and non-Abelian gauge anomaly cancellation. It remains to be seen if similar results can be obtained in heterotic $T^6/\mathbb{Z}_N$ and $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ compactifications to four dimensions. It will also be interesting to see if the heterotic non-Abelian orbifolds have analogous matchings to embeddings with higher rank $\mathrm{U}(n)$ bundles.

Last but not least, for the SO(32) heterotic orbifolds listed here, threshold corrections analogous to the ones computed for $E_8 \times E_8$ in [18] as $\mathcal{N} = 2$ sectors in four dimensions

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might be evaluated and the moduli dependence beyond the leading order in the geometric regime on Calabi-Yau three-folds extracted.

Acknowledgments

We are grateful for the hospitality of the workshop “New Directions beyond the Standard Model in Field and String Theory” at the the Galileo Galilei Institute, Florence, where this work was initiated.
G.H. thanks R. Blumenhagen, F. Gmeiner, S. Stieberger and T. Weigand for discussions and K. Wendland for a useful communication; M.T. thanks S. Groot Nibbelink and A. Hebecker for discussions.

A. Decomposition of representations upon gauge symmetry breaking

In this appendix, we list the types of gauge symmetry breaking required in order to compare the orbifold spectra with the smooth $K3$ compactifications. The blow-ups of orbifold singularities trigger the gauge symmetry breaking needed.

$$
\begin{array}{ll}
\mathrm{SU}(N+M) & \rightarrow \mathrm{SU}(N)\times\mathrm{SU}(M)\times\mathrm{U}(1) \\
\hline
\mathbf{N}+\mathbf{M} & \rightarrow (\mathbf{N},\mathbf{1})_M + (\mathbf{1},\mathbf{M})_{-N} \\
\hline
\frac{\mathbf{(N+M)(N+M-1)}}{\mathbf{2}} & \rightarrow (\mathbf{N},\mathbf{M})_{-N+M} + \left(\frac{\mathbf{N(N-1)}}{\mathbf{2}},\mathbf{1}\right)_{2M} + \left(\mathbf{1},\frac{\mathbf{M(M-1)}}{\mathbf{2}}\right)_{-2N} \\
\hline
(\mathbf{N}+\mathbf{M})^{\mathbf{2}}-\mathbf{1} & \rightarrow (\mathbf{N},\overline{\mathbf{M}})_{N+M} + (\overline{\mathbf{N}},\mathbf{M})_{-N-M} \\
 & \quad +(\mathbf{1},\mathbf{1})_0 + (\mathbf{N}^{\mathbf{2}}-\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{M}^{\mathbf{2}}-\mathbf{1})_0 \\
\hline
\frac{\mathbf{(N+M)(N+M-1)(N+M-2)}}{\mathbf{6}} & \rightarrow \left(\frac{\mathbf{N(N-1)(N-2)}}{\mathbf{6}},\mathbf{1}\right)_{3M} + \left(\mathbf{1},\frac{\mathbf{M(M-1)(M-2)}}{\mathbf{6}}\right)_{-3N} \\
 & \quad + \left(\frac{\mathbf{N(N-1)}}{\mathbf{2}},\mathbf{M}\right)_{2M-N} + \left(\mathbf{N},\frac{\mathbf{M(M-1)}}{\mathbf{2}}\right)_{M-2N}
\end{array}
$$

The most frequent case $\mathrm{SU}(N+1)\rightarrow\mathrm{SU}(N)\times\mathrm{U}(1)$ is obtained by setting $M=1$ in the above breaking pattern.

$$
\begin{array}{ll}
\mathrm{SO}(2M) & \rightarrow \mathrm{SU}(M)\times\mathrm{U}(1) \\
\hline
\mathbf{2M} & \rightarrow \mathbf{M}_1 + \overline{\mathbf{M}}_{-1} \\
\hline
\mathbf{Adj}^{\mathrm{SO}(2M)} & \rightarrow \mathbf{Adj}_0^{\mathrm{SU}(M)} + \mathbf{1}_0 + [\mathbf{Anti}_2^{\mathrm{SU}(M)} + c.c.] \\
\hline
\mathbf{2}_{\pm}^{M-1} & \rightarrow \left\{ \begin{array}{c} \sum_{k=0}^{[M/2]} \begin{pmatrix} M \\ 2k \end{pmatrix}_{-M/2+2k} \\ \sum_{k=0}^{[(M-1)/2]} \begin{pmatrix} M \\ 2k+1 \end{pmatrix}_{-M/2+2k+1} \end{array} \right.
\end{array}
$$

In the decomposition of the spinor representations, we have used the notation

$$\begin{pmatrix} M \\ 0 \end{pmatrix} = \mathbf{1}, \quad \begin{pmatrix} M \\ 1 \end{pmatrix} = \mathbf{M}, \quad \begin{pmatrix} M \\ 2 \end{pmatrix} = \mathbf{Anti}^{\mathrm{SU}(M)}, \ldots$$

$$\ldots, \begin{pmatrix} M \\ M-2 \end{pmatrix} = \overline{\mathbf{Anti}}^{\mathrm{SU}(M)}, \quad \begin{pmatrix} M \\ M-1 \end{pmatrix} = \overline{\mathbf{M}}, \quad \begin{pmatrix} M \\ M \end{pmatrix} = \mathbf{1},$$

for the antisymmetric tensors of order $2k$ and $2k+1$.

Furthermore, in order to compute the U(1) embeddings inside an E_8 gauge factor, we need the following decompositions:

$\mathrm{SO}(2N)$	$\to \mathrm{SO}(2N-2) \times \mathrm{U}(1)$
$(\mathbf{2N})$	$\to (\mathbf{2N-2})_0 + [(\mathbf{1})_2 + c.c.]$
$\left(\frac{\mathbf{2N(2N-1)}}{\mathbf{2}}\right)$	$\to (\frac{\mathbf{(2N-2)(2N-3)}}{\mathbf{2}})_0 + (\mathbf{1})_0$ $+[(\mathbf{2N-2})_2 + c.c.]$
$\left(\mathbf{2}_{\pm}^{N-1}\right)$	$\to \left(\mathbf{2}_{\pm}^{N-2}\right)_1 + \left(\mathbf{2}_{\mp}^{N-2}\right)_{-1}$

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